Introduction to the Fixed Income Markets

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MSc Financial Mathematics
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SMM269 Fixed Income

Academic Year 2019-20

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Outline

Why a course on Fixed Income?

Complexity of Fixed Income Markets

The Size of the Debt Market

Conclusions

Reading List

Required Readings

- Veronesi P., (2010). Fixed Income Securities. Chapter **1**.
- Bank of International Settlements. `http://www.bis.org/`.

Why a course on Fixed Income?

Introduction

Why model interest rates?

- Find the time value of the money:
- how much would an individual be willing to pay today for a promised value of one euro that is promised for a known future date?
- Pricing of interest derivative instruments:
    1. Bonds
    2. Bond options
    3. Bond futures
    4. Interest rate swaps
    5. Swaptions
    6. Caps, Floors
- Risk Management in the Fixed Income arena
- Consider that the bond market has a greater extension than the stock market.

Basic Valuation Problems

- `Valuation` of payments that are promised for a future specified date
    - Example: valuation of a structured bond promising payments with the amount and date of the payments determined by the
        - face value
        - maturity date
        - reference rate
        - coupon formula: fixed, floating, with optionalities, etc.
- `Hedging` a position in the fixed income market.
    - Example: how shifts in the market interest rates can affect the value of your book? How can we devise an hedging strategy?
- `Interest rate modelling:` in order to price and hedge interest rate derivatives we need to model the dynamics of the term structure of interest rates.

Complexity of Fixed Income Markets

Complexity of Fixed Income Markets

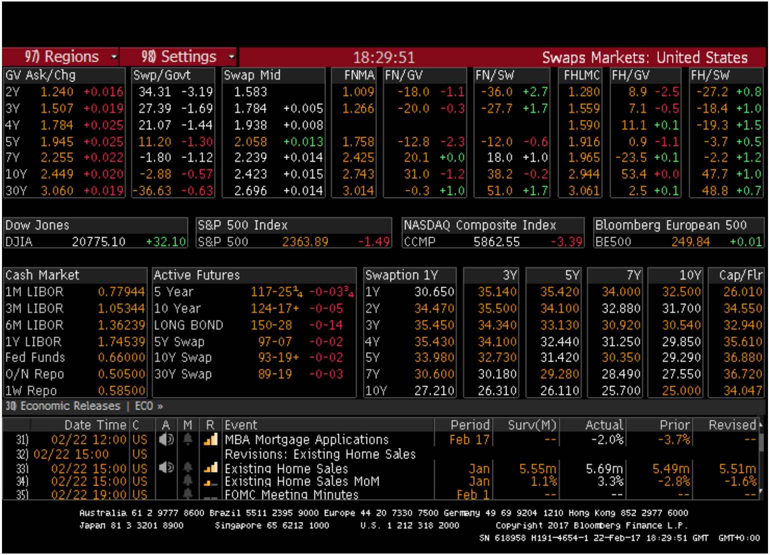

Figure 1: Source: Bloomberg Screen BTMM, February 23,2017.

The Size of the Debt Market

The Growth in Debt Market Size

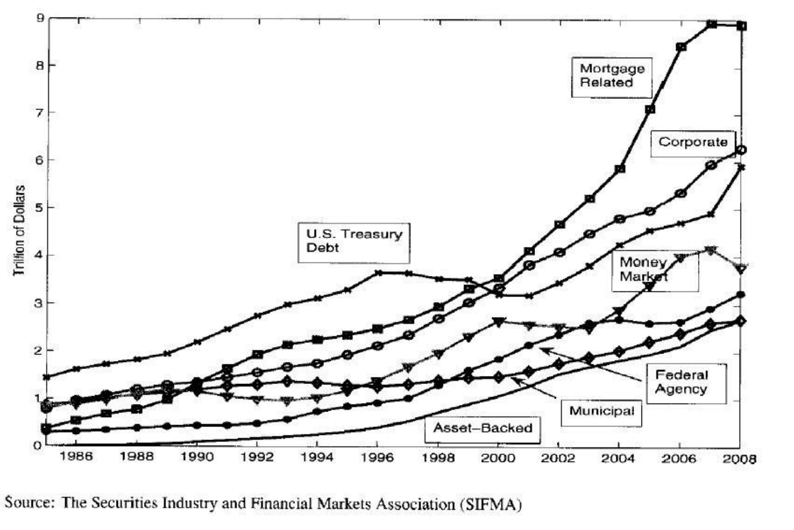

Figure 2: Veronesi (2010).

Global Debt Market Size

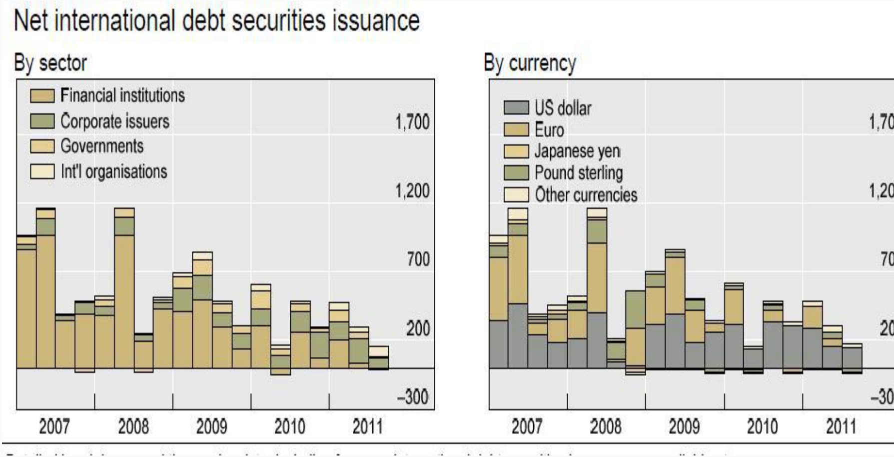

Figure 3: BIS Quarterly Review, December 2011.

The Growth in Derivatives Markets

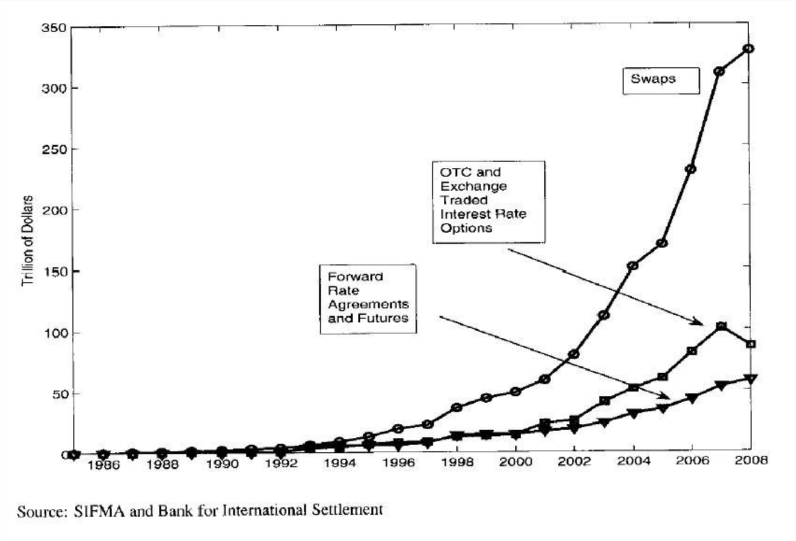

Figure 4: Outstanding Notionals. Source: Veronesi (2010).

Global OTC Derivatives

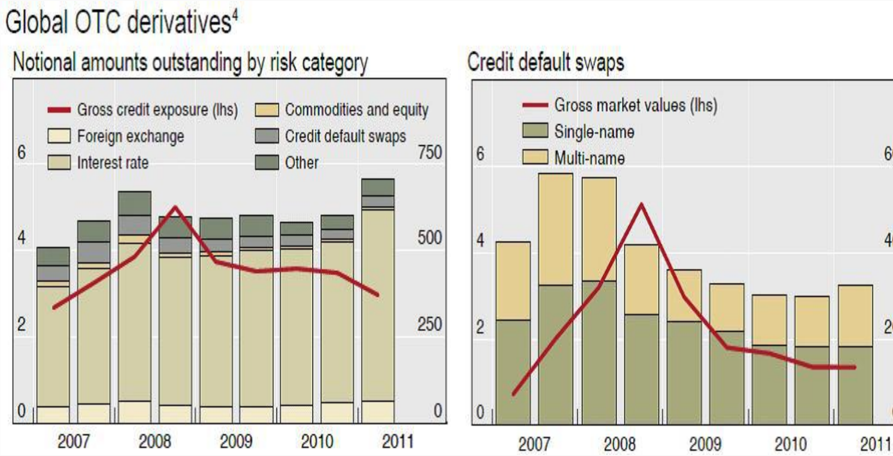

Figure 5: BIS Quarterly Review, December 2011.

Conclusions

Conclusions

- We have given an idea about the complexity of the fixed income world
- The aim of the course will be to discuss the most important pricing models and hedging techniques and how they deal with market data, conventions and products.

Yield Curve Basics: a Taxonomy

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References

Required Readings

- Veronesi. Chapters 1 and 2.

Useful Readings

- OpenGamma Documentation. Bond Pricing, by March Henrard. https://developers.opengamma.com/quantitative-research/Bond-Pricing-OpenGamma.pdf
- Bond Markets: Structures and Yield Calculations by Patrick J Brown , 1998 International Securities Market Association.
- Frequently Asked Questions about Euribor. Available at https://www.emmi-benchmarks.eu/assets/files/Euribor%20FAQs%20Final.pdf.
- LIBOR: Frequently Asked Questions. Available at https://www.theice.com/publicdocs/IBA_LIBOR_FAQ.pdf

Excel Files

- FI_BasicYields.xlsm

Outline

Basic Concepts: Zero-Coupon Bond

In BS 1973, the price of the underlying is given by the market S_0

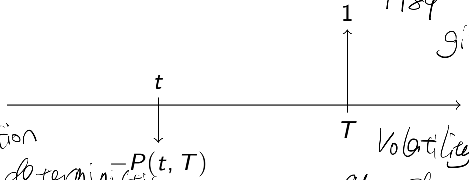
* the price of the underlying is given by ↗ Merton 74 model: ← consistency problem

$dr = \mu dt + \sigma dW_t$

- Zero-Coupon bond: Call option on ZCB $\Pi = (P(T_1, T_2) - K)^+$
  - it is a contract that guarantees to its holder the payment of one unit of currency at time T, with no intermediate payments.

Ho & Lee 1989 $P(t,T)$ given by the market

Model spot rate
Model DF
Model FWD rate



BS for stock option assume r to be deterministic

Volatility → 0 as $\tau \to 0$

  - $P(t, T)$ is the value at time $t < T$ of unit of currency due in T.
  - We have $P(T, T) = 1$, for all T.
  - $P(t, T)$ is also called the t-price of a zero-coupon bond (expiring in T).

use GBM time-dependent σ_t

Fixed payoff ← different Stock option

instantaneous rate

$$\mathbb{E}\left[e^{-\int_s^t r(u) du} \Big| \mathcal{F}_t \right]$$

Term Structure of Discount Factors

- If we consider the zcb price as function of time to maturity

$$\tau \to P(t, t+\tau),$$

we have the discount function.

- The plot of $P(t, t+\tau)$ against τ is called the `term structure of discount factors`.
- $P(t, t+\tau)$ used to be a monotonic decreasing function. It's no true anymore.

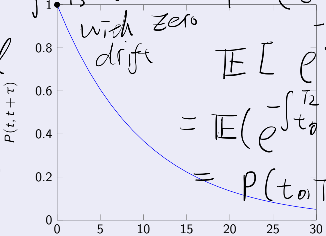


Figure 1: The Discount Function

return: $\dfrac{1 - P(t,T)}{\alpha \; P(t,T)}$ simple annual return

$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) + \mathrm{Cov}(X,Y)$

$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}^*(Y)$ → a new measure

f is a G.B.M with zero drift

$\Pi : (f(t_1, T_1, T_2) - K)^+ \alpha(T_1, T_2)$

$\mathbb{E}\left[e^{-\int_{t_1}^{T_2} r_s ds} \Pi_{t_1} \mid \mathcal{F}_{t_1} \right]$

$= \mathbb{E}\left(e^{-\int_{t_0}^{T_2} r_s ds}\right) \mathbb{E}^*(\Pi_{t_1})\, \alpha$

$= P(t_0, T_2)\{ f(t, T_1, T_2) N(d_1) - K N(d_2) \}$ BS with $r = 0$

forward rate is a martingale

Computing Present and Final value

Using zcb prices we can compute present values and final values.

Discounting: Present Value

Compute the present value in t of the notional N that will be received in T

t	T
$-P(t, T) \times N$	$+N$

Compounding: Final Value

Compute the value in T of the capital C that is invested in t

t	T
$-C$	$\frac{C}{P(t,T)}$

Present value of a sequence of cash flows I

- How do do we determine the present value of a sequence of cash flows?

t	T_1	T_2	T_3	$\cdots$	T_n
PV=?	c_1	c_2	c_3	$\cdots$	c_n

- We use the linearity of market prices: the price of a basket of assets is equal to the sum of the prices of the individual components.

Discounting: Present Value of a sequence of cash flows

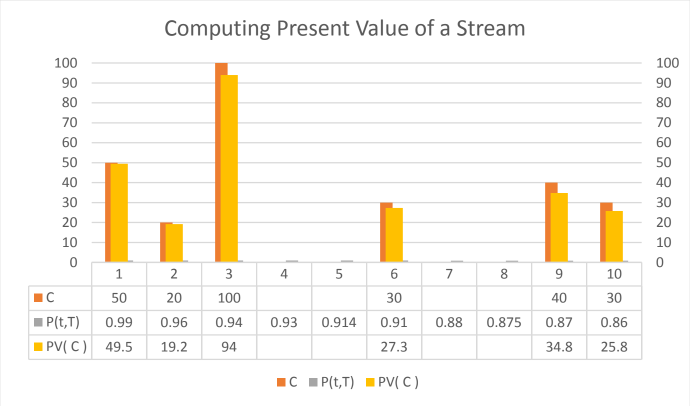
The present value in t of the sequence of known cash flows $\mathbf{c} = [c_1, \cdots, c_n]$ that will be available at future known dates $\mathbf{T} = [T_1, \cdots, T_n]$ is

$$PV(\mathbf{c}, \mathbf{T}) = \sum_{i=1}^{n} c_i \times P(t, T_i).$$

Example (Computing the present value of a stream of cash flows)

Table 1: The present value of the amounts in column 2 that will be received at times in column 1 is given in column 4. The present values are computed using the term structure of discount factors in column 3. Summing up the present values we obtain 252.4. To receive this amount today is financially equivalent to receive the cash flows at future dates.

$T(years)$	C	$P(t,T)$	$PV(C)$
1	50	0.99	49.5
2	20	0.96	19.2
3	100	0.94	94
6	30	0.91	27.3
9	40	0.87	34.8
10	30	0.86	25.8
		Sum	**252.4**



	1	2	3	4	5	6	7	8	9	10
C	50	20	100			30			40	30
P(t,T)	0.99	0.96	0.94	0.93	0.914	0.91	0.88	0.875	0.87	0.86
PV(C)	49.5	19.2	94			27.3			34.8	25.8

Figure 2: Computing the present value of a stream of cash flows.

Spot rates I

Spot rates

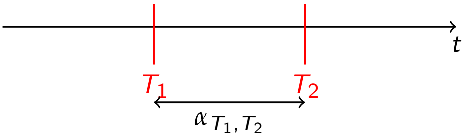
A spot rate (or zero rate) for maturity T is the rate of return related to investing in a zero-coupon bond, i.e. on an investment, starting at time t, providing a unique payment at T.

To convert zcb prices into rates, two issues:

1. the **compounding frequency**: simple compounding, periodic/annual/continuous compounding, discount convention.
2. The **day-count convention**: defining how time between two dates is computed.
   - The quantity $\alpha_{t,T}$ measures the year fraction between t and T computed according to a given convention (e.g. ACT/360, ACT/ACT, 30/360).
   - For example, if we use the ACT/360 convention and there are 47 days between t and T, then

$$\alpha_{t,T} = \frac{47}{360} = 0.13055.$$

The accrual factor α_{T_1,T_2} and the day count convention



Day Count Convention

- The day count convention is defined as the way in which the ratio of the number of days between coupon dates to the number of days in the reference period (e.g., year) is calculated.
- The most common day conventions for fixed rate bonds are
  - A day count convention of actual days to maturity to actual days in the year (ACT/ACT).
  - A day count convention of 30-day months to maturity to a 360 days in the year (30/360).
- The numerator tells you how to calculate the number of days between two successive coupon dates, the denominator tells you how to calculate the number of days in the year:
  - Act means the actual number of calendar days,
  - 360 means "assume there are 360 days in a year", and
  - 30 means "assume there are 30 days a month".
- Also note: If a coupon date falls on a non-trading day, the coupon is moved to a trading day following a certain convention. The actual amount paid on the coupon date can be or cannot be then modified to account for it.

Example (The day count convention 30/360 (ISDA))

- The 30/360 method group a certain number of methods that have in common the accrual factor as

$$\frac{360 \times (Y_2 - Y_1) + 30 \times (M_2 - M_1) + (D_2 - D_1)}{360}$$

but differs on how the Y_i, M_i and D_i are computed.

- The year fraction between January 31st, 2017 and February 28th, 2020 is:

$$D_1 = 30, D_2 = 28; M_1 = 1, M_2 = 2; Y_1 = 2017, Y_2 = 2020,$$

therefore

$$\frac{360 \times (2020 - 2017) + 30 \times (2 - 1) + (28 - 30)}{360} = \frac{1108}{360} = 3.07777.$$

Example (The day count convention ACT/360)

- A year is 360 days long and the year fraction between two dates is the actual number of days between them divided by 360.
- The accrual factor is the actual number of accrued days divided by 365:

$$\frac{D_2 - D_1}{360}$$

  where $D_2 - D_1$ is the number of days between the two dates.
- The year fraction between January 31st, 2017 and February 28th, 2020 is:

$$D_2 = 43889(usingExcel), D_1 = 42766$$

  therefore

$$\frac{43889 - 42766}{360} = \frac{1123}{360} = 3.11944.$$

Compounding Conventions and Spot Rates

Convention	Spot rate	Compounding frequency	Use
Simple	$L(t, T)$	1	Money Market
Compound	$Y(t, T)$	1	YTM for bonds
Compound	$Y_m(t, T)$	m	YTM for US bonds
Continuous	$R(t, T)$	∞	Modelling
Instantaneous	$r(t)$		Modelling
Discount	$d(t, T)$	1	US T-Bills

From Discount Factors to Spot Rates

Table 2: From Discount Factors to Spot Rates: Common conventions for computing returns on zcb. Examples are made assuming: $P = 0.98, \alpha_{t,T} = 0.25, m = 4$.

Compounding	Formula	Example
Simple	$L(t,T) = \frac{1}{\alpha_{t,T}} \frac{1-P(t,T)}{P(t,T)}$	$\frac{1}{0.25} \frac{1-0.98}{0.98} = 8.1633\%$
Periodic	$Y_m(t,T) = m\left(\left(\frac{1}{P(t,T)}\right)^{\frac{1}{m\times\alpha_{t,T}}} - 1\right)$	$4\left(\left(\frac{1}{0.98}\right)^{\frac{1}{4\times 0.25}} - 1\right) = 8.1632\%$
Annually	$Y(t,T) = \left(\frac{1}{P(t,T)}\right)^{\frac{1}{\alpha_{t,T}}} - 1$	$\left(\frac{1}{0.98}\right)^{\frac{1}{0.25}} - 1 = 8.4165\%$
Continuous	$R(t,T) = -\frac{\ln P(t,T)}{\alpha_{t,T}}$	$-\frac{\ln(0.98)}{0.25} = 8.081\%$
Instantaneous	$r(t) = \lim_{T\to t} R(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}\Big\vert_{T=t}$	
Discount	$d(t,T) = \frac{1-P(t,T)}{\alpha_{t,T}}$	$\frac{1-0.98}{0.25} = 8\%$

From Spot Rates to Discount Factors

Table 3: From Spot Rates to Discount Factors: Common conventions for computing zcb prices from spot rates

Compounding	Formula	Example
Simple	$P(t,T) = \frac{1}{1+L(t,T)\alpha_{t,T}}$	$\frac{1}{1+8.1633\%\times 0.25} = 0.98$
Periodic	$P(t,T) = \frac{1}{\left(1+\frac{Y_m}{m}(t,T)\right)^{m\times\alpha_{t,T}}}$	$\frac{1}{\left(1+\frac{8.1632\%}{4}\right)^{4\times 0.25}} = 0.98$
Annually	$P(t,T) = \frac{1}{(1+Y(t,T))^{\alpha_{t,T}}}$	$\frac{1}{(1+8.4165\%)^{0.25}} = 0.98$
Continuous	$P(t,T) = e^{-R(t,T)\alpha_{t,T}}$	$e^{-8.081\%\times 0.25} = 0.98$
Instantaneous	$P(t,T) = \mathbb{E}_t\left(e^{-\int_t^T r(s)ds}\right)$	**CAREFUL!**
Discount	$P(t,T) = 1 - d(t,T)\,\alpha_{t,T}$	$1 - 8\% \times 0.25 = 0.98$

Computing Present value (discounting) using spot rates

- `Discounting`: compute the present value in t of the notional N that will be received in T

$$PV(N) = N \times P(t, T).$$

  - Simple convention

$$PV(N) = \frac{N}{1 + L(t, T)\,\alpha_{t,T}}.$$

  - Annually compounded

$$PV(N) = \frac{N}{(1 + Y(t, T))^{\alpha_{t,T}}}.$$

  - Continuously compounded

$$PV(N) = N \times e^{-\alpha_{t,T} \times R(t,T)}.$$

  - Discount convention

$$PV(N) = N \times (1 - d(t, T) \times \alpha_{t,T}).$$

Computing Final value (compounding) using Spot Rates

- `Compounding`: compute the value in T of the capital C that is invested in t

$$FV(N) = \frac{N}{P(t,T)}.$$

  - Simple convention

$$FV(C) = C \times (1 + L(t,T)\,\alpha_{t,T}).$$

  - Annually compounded

$$FV(C) = C \times (1 + Y(t,T))^{\alpha_{t,T}}.$$

  - Continuously compounded

$$FV(C) = C \times e^{\alpha_{t,T} \times R(t,T)}.$$

  - Discount convention

$$FV(C) = \frac{C}{(1 - d(t,T) \times \alpha_{t,T})}.$$

Term Structure of Spot Rates I

- It is the graph of the function mapping maturities into spot rates at time t: it associates at each maturity the value of the spot rate from time t to the desired maturity:

$$\{\mathbf{T_i} - t, \text{Spot Rate}(t, \mathbf{T_i})\}_{i=1,\cdots,n}.$$

- The practice is to plot a set of simple rates versus time to maturity up to 1 year and then to plot annually compounded or continuously spot rates.

Term Structure of Spot Rates II

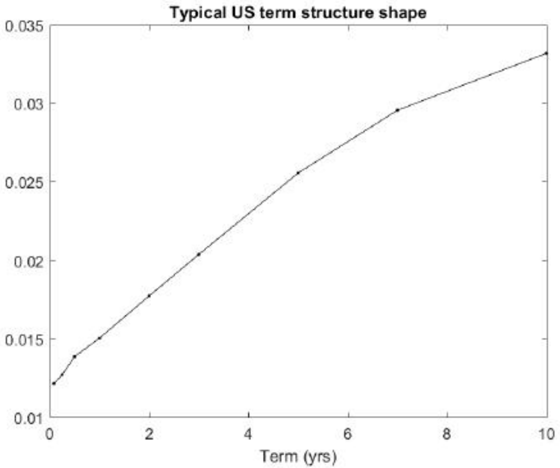


Figure 3: Typical US term structure shape for different terms (1m-10Y) in the period 1st Aug. 2001 to 24th Jan. 2018

Typical Shapes of the Term Structure of Spot Rates

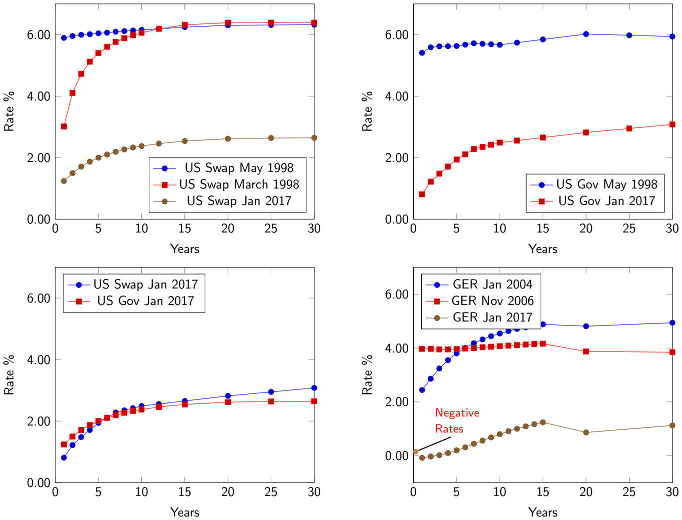


Table 4: Top Left: US Swap Curve. **Top Right**: US Govt. **Bottom Right**: US Govt vs Swap Curve. **Bottom Left**: Germany Spot Curve.

The Term Structure moves in time

- The important point is that a term structure changes over time.
- This means that the present value of a bond will change over time, i.e. we will be interested not only to measure the value of a position but to understand how it changes over time.
- The most common changes in the term structure shape are changes in level, slope and curvature.

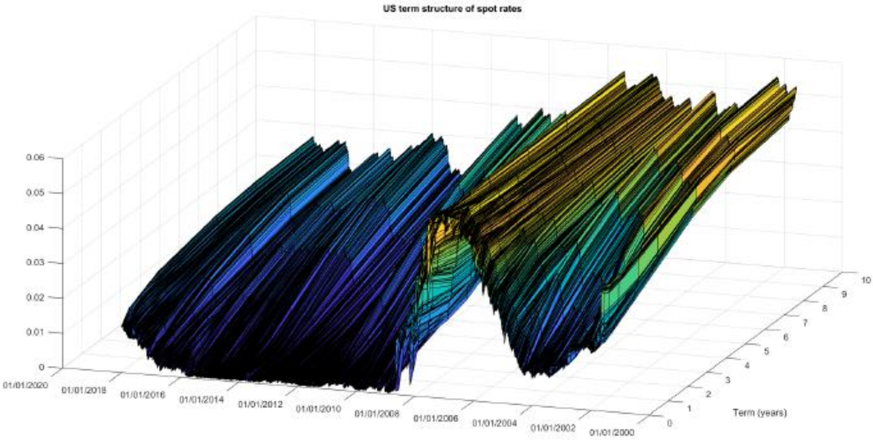


Figure 4: US term structure for different terms (1m-10Y) in the period 1st Aug. 2001 to 24th Jan. 2018. Go to https://youtu.be/oJhUY9ZT1CI to see the US term structure movie.

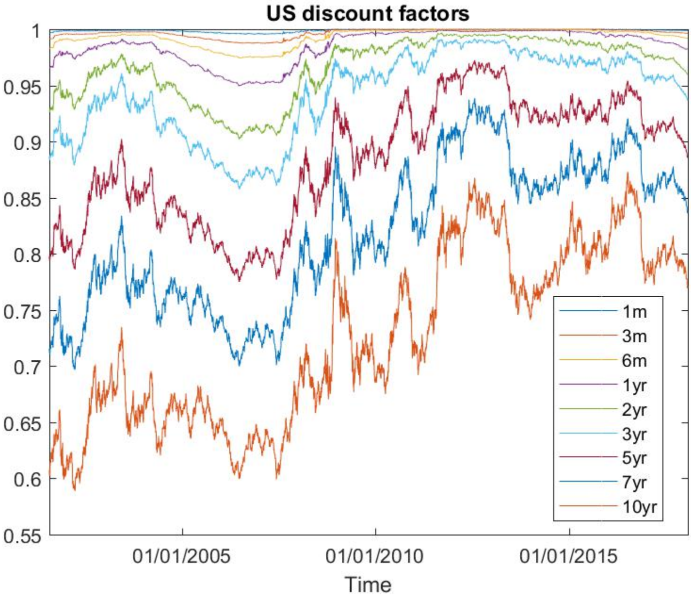


Figure 5: US discount factors over time: the discount factors are not constant and they are more volatile longer the tenor.

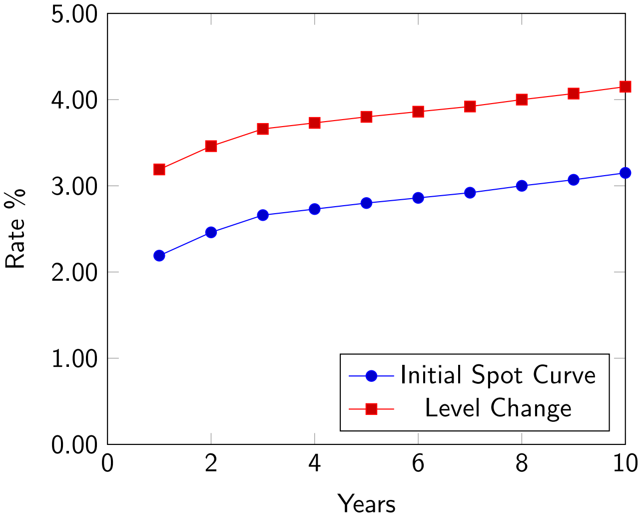


Figure 6: Change in the level of the term structure

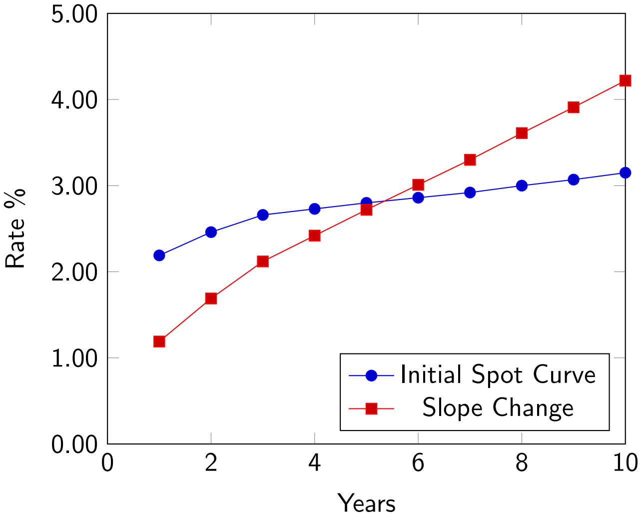


Figure 7: Change in the slope of the term structure

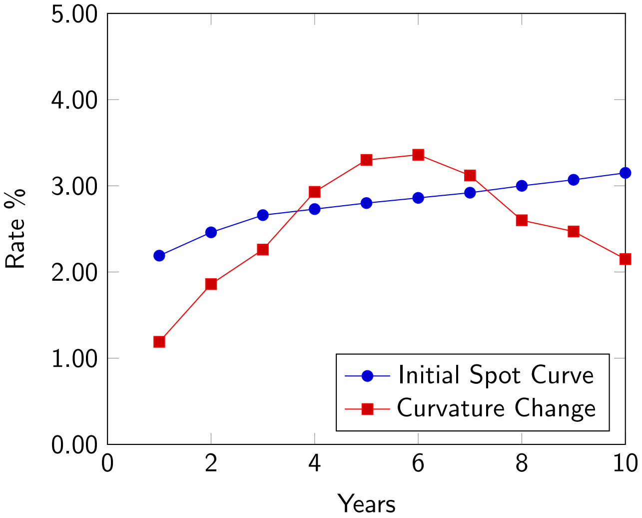


Figure 8: Change in the curvature of the term structure

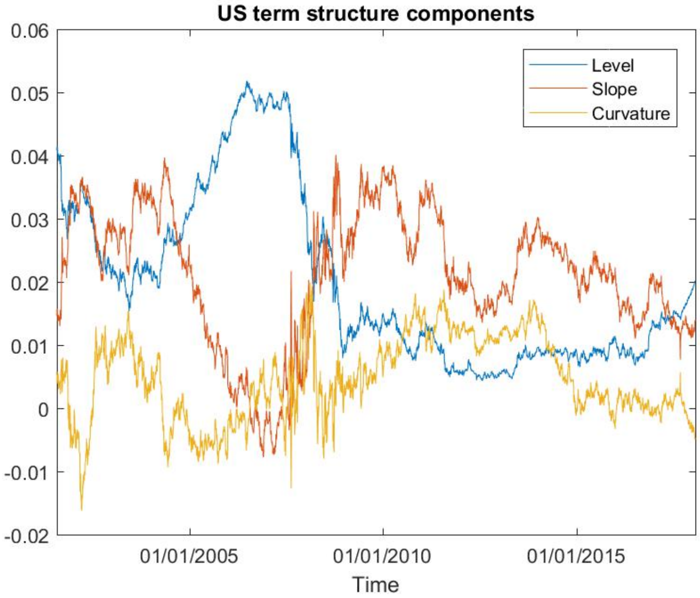


Figure 9: Typical US term structure factors. **Level**: average yield across maturities; **Slope**: long term yield - short term yield; **Curvature**: Long Term Yield+Short Term Yield- 2x Mid Term Yield

- In a given day:
  - the **level** is estimated taking the average of the spot rates

Model symplicity & Consistency

$$level(t) = \frac{1}{n}\sum_{i=1}^{n} R(t, T_i)$$

  - the **slope** is estimated taking the difference between a long term and a short term rate

$$slope(t) = R(t, T_n) - R(t, T_1)$$

    where for example, T_1= 1 yr and T_n =30 yrs.
  - the **curvature** is estimated by the following quantity

$$curvature(t) = R(t, T_n) - 2R(t, T_m) + R(t, T_1)$$

    where for example, T_m= 10 or 15 yrs.

Table 5: Some statistics (US market)

	Level (%)	Slope (%)	Curvature (%)
Mean	2.0017	2.1045	0.4586
Std. Dev.	1.3616	1.04	0.6531
Maximum	5.1856	4.01	1.99
Minimum	0.4322	-0.78	-1.62

Some questions

- Some questions:
  1. Where do we get the discount factors $P(t, T)$ (or the spot rates) from?
     - ⋆ We distinguish market and implied curve: usually, the first ones are read directly from the market, the others are constructed using market data.
     - ⋆ Besides we can define a curve for each different issuer/rating, that determines the term structure of credit spreads.
  2. We start by examining the following markets
     - ⋆ Ibor-like markets: it provides quotations of simple spot rates up to 12 months.
     - ⋆ ZCB markets that provide direct information about the discount term structure.
     - ⋆ Fixed rate bond markets that provides information that needs to be someway elaborated to produce the discount term structure.
  3. How we model term structure changes over time?
     - ⋆ We need a term structure model
     - ⋆ A model requires a quantity to be modelled.
     - ⋆ Depending on the choice we can build different models: short rate models, (instantaneous) forward rate models, (discrete) forward rate models.

Market Quotation: Ibor-like indexes (Simple Interest Rates)

Par rate = swap rate

* LIBOR can be manipulated

Ibor-like indexes

- Ibor-like indexes are indexes related to interbank lending between one day and one year.
- The rates are banks' estimates but usually do not refer to actual transactions.
- The so called contributor banks are of first class market standing and they have been selected to ensure that the diversity of the money market is adequately reflected.
- LIBOR is used as reference rate for bonds, derivatives (swaps and caps/floors), mortgages and so on. This has created a conflict of interest in the past and fraudulent actions connected to the determination of the LIBOR, the so called **LIBOR scandal**.
- In the US market the setting procedure of the LIBOR rates has been recently changed. SOFR is now based on transactions in the Treasury repurchase market, where banks and investors borrow or loan Treasuries overnight.
- A similar reform is underway in the Euro market, with the ESTER rate replacing in the near future the EURIBOR rate.
- Still a lot of open problems.

How Libor is fixed

- Ibor rates are published once per day and are usually computed as the trimmed average between rates contributed by participating banks.
  - In the Euro zone, for each maturity, the highest and lowest 15% of all the quotes collected are eliminated. The remaining rates are averaged and rounded to three decimal places. The resulting rate is then published to the market at approximately 11.00 am Frankfurt time.
  - In the ICE LIBOR, the highest and lowest 25% are removed and the rest is averaged (the actual number of banks removed depends on the number of submi tters for each currency). The resulting rate is then published to the market at approximately 11.45 am London time.
  - JBA TIBOR (Tokyo InterBank Offered Rate) excludes the top two and the bottom two reference rates for each maturity and takes the average of the remaining rates.

Table 6: The panel of contributing banks currently (January 2020) consists of 18 CONTRIBUTORS.

Belgium	**Luxembourg**
Belfius	Banque et Caisse d'Épargne de l'État
France	**Netherlands**
BNP-Paribas	ING Bank
HSBC France	**Portugal**
Natixis	Caixa Geral De Depósitos (CGD)
Crédit Agricole s.a.	**Spain**
Société Générale	Banco Bilbao Vizcaya Argentaria
Germany	Banco Santander
Deutsche Bank	CECABANK
DZ Bank	CaixaBank S.A.
Italy	**UK**
Intesa Sanpaolo	Barclays
UniCredit	

Table 7: Quotation mechanism of Euribor (30th Jan 2018). Excel file: FI_BasicYields, Sheet: Computing LIBOR

	1 Week	2 Weeks	1 Month	2 Months	3 Months	6 Months	9 Months	12 Months
BNP-Paribas	-0.41	-0.42	-0.41	-0.40	-0.35	-0.34	-0.27	-0.26
Monte Dei Paschi Di Siena	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
Bilbao Vizcaya Argentaria	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
Banco Santander	-0.38	-0.37	-0.37	-0.34	-0.33	-0.27	-0.22	-0.19
Caisse d'Épargne de l'État	-0.38	-0.38	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
Barclays Bank	-0.45	-0.43	-0.40	-0.38	-0.37	-0.34	-0.31	-0.28
Belfius	-0.37	-0.37	-0.37	-0.35	-0.33	-0.28	-0.23	-0.20
CECABANK	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
Caixa Geral De Depósitos	-0.37	-0.36	-0.36	-0.33	-0.31	-0.26	-0.20	-0.18
CaixaBank S.A.	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
Crédit Agricole s.a.	-0.38	-0.37	-0.36	-0.34	-0.32	-0.30	-0.26	-0.22
DZ Bank	-0.39	-0.39	-0.38	-0.35	-0.32	-0.26	-0.20	-0.14
Deutsche Bank	-0.37	-0.36	-0.31	-0.31	-0.31	-0.19	-0.15	-0.11
HSBC France	-0.37	-0.37	-0.37	-0.34	-0.33	-0.27	-0.21	-0.18
ING Bank	-0.36	-0.36	-0.36	-0.34	-0.33	-0.28	-0.23	-0.19
Intesa Sanpaolo	-0.38	-0.37	-0.37	-0.34	-0.33	-0.27	-0.22	-0.19
National Bank of Greece	-0.37	-0.37	-0.37	-0.34	-0.32	-0.27	-0.21	-0.18
Natixis	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
Société Générale	-0.36	-0.35	-0.36	-0.29	-0.28	-0.33	-0.28	-0.26
UniCredit	-0.38	-0.38	-0.38	-0.34	-0.33	-0.27	-0.21	-0.18
Trimmed Average	-0.377	-0.371	-0.369	-0.341	-0.328	-0.278	-0.222	-0.191

Table 8: Computing EURIBOR from bank contributions

Sorted Quotes	1 Week	2 Weeks	1 Month	2 Months	3 Months	6 Months	9 Months	12 Months
1	-0.45	-0.43	-0.41	-0.40	-0.37	-0.34	-0.31	-0.28
2	-0.41	-0.42	-0.40	-0.38	-0.35	-0.34	-0.28	-0.26
3	-0.39	-0.39	-0.38	-0.35	-0.33	-0.33	-0.27	-0.26
4	-0.38	-0.38	-0.38	-0.35	-0.33	-0.30	-0.26	-0.22
5	-0.38	-0.38	-0.37	-0.34	-0.33	-0.28	-0.23	-0.20
6	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.23	-0.19
7	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
8	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
9	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
10	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
11	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
12	-0.38	-0.37	-0.37	-0.34	-0.33	-0.28	-0.22	-0.19
13	-0.38	-0.37	-0.37	-0.34	-0.33	-0.27	-0.22	-0.19
14	-0.37	-0.37	-0.37	-0.34	-0.33	-0.27	-0.22	-0.19
15	-0.37	-0.37	-0.37	-0.34	-0.32	-0.27	-0.21	-0.18
16	-0.37	-0.37	-0.36	-0.34	-0.32	-0.27	-0.21	-0.18
17	-0.37	-0.36	-0.36	-0.34	-0.32	-0.27	-0.21	-0.18
18	-0.37	-0.36	-0.36	-0.33	-0.31	-0.26	-0.20	-0.18
19	-0.36	-0.36	-0.36	-0.31	-0.31	-0.26	-0.20	-0.14
20	-0.36	-0.35	-0.31	-0.29	-0.28	-0.19	-0.15	-0.11
Trimmed Average	-0.377	-0.371	-0.369	-0.341	-0.328	-0.278	-0.222	-0.191

the underlying is unsecured
i.e. counterparty risk exists

Useful websites

- Useful information available at
  https://www.theice.com/iba/libor
  http://www.euribor-ebf.eu/euribor-org/about-euribor.html
  https://www.emmi-benchmarks.eu/euribor-org/faq.html#faq2
  http://www.jbatibor.or.jp/english/about/
  https://en.wikipedia.org/wiki/Libor_scandal

IBOR conventions I

- The IBOR conventions are the same for all currencies.
  1. IBOR rates follow the simple compounding convention.
  2. For all currencies other than EUR and GBP the period between Fixing Date and Value Date will be two London business days after the Fixing Date.
  3. However, if that day is not both a London business day and a business day in the principal financial center of the currency concerned, the next following day that is a business day in both centers shall be the Value Date.
  4. The business day convention is modified following [1] and the end-of-month rule applies.
  5. For all currencies except GBP, the day-count convention is ACT/360.
  6. For GBP, the Fixing Date and Value Date are the same (0 day spot lag).

IBOR conventions II

- Some Ibor-like indexes and their main characteristics are summarized in the following Table.

Currency	Name	Maturities	Convention	Spot Lag	Bloomberg
CHF	LIBOR	O/N–12M	ACT/360	2	SF00xxx
EUR	EURIBOR	1W–12M	ACT/360	2	EUR0xxx
EUR	EURIBOR	1W–12M	ACT/365	2	
EUR	LIBOR	O/N–12M	ACT/360	2	EU00xxx
GBP	LIBOR	O/N–12M	ACT/365	0	BP00xxx
JPY	LIBOR	O/N–12M	ACT/360	2	JY00xxx
JPY	Japan TIBOR	1W–12M	ACT/365	2	
JPY	Euroyen TIBOR	1W–12M	ACT/360	2	
USD	LIBOR	O/N–12M	ACT/360	2	US00xxx

Table 9: Ibor-like indexes for the main currencies. In the Bloomberg code, the xxx should be replaced by the tenor (T/N, 01W, 11M, etc.) and followed by Index.

[1] Dates are adjusted to the next good business day unless that day falls in the next calendar month in which case the date is adjusted to the previous good business day.

Tenor, Accrual Factor and Day Count Convention I

Tenor The time length of the investment is called tenor

- A deposit starting today and ending in 3 months has a 3 months tenor.

Spot Lag

- The number of business days between the Trade Date and the Value Date.
- Money changes hands on the Value Date.
- Exception is the Overnight (O/N) deposit. Money changes hands on the Trade Date.

Day Count Convention The year fraction between two dates is computed according to a particular day count convention, eg. ACT/360.

- ACT/360 means that we have to divide the effective number of days between starting and ending date by 360.

Tenor, Accrual Factor and Day Count Convention II

Accrual Factor The year fraction between two dates.

- If the effective number of days between two dates is 92 and the day count convention is ACT/360, the accrual factor turns to be

$$\frac{92}{360}$$

- If the effective number of days between two dates is 92 and the day count convention is ACT/365, the accrual factor turns to be

$$\frac{92}{365}$$

End of Month If the 1 month quote for 28 February intended for a money market deposit maturing on 28 March or on 31 March (assuming obviously they are business days)? The EOM rule says that the maturity date for this will be 31 March and not 28 March.

LIBOR and Euribor Quotes

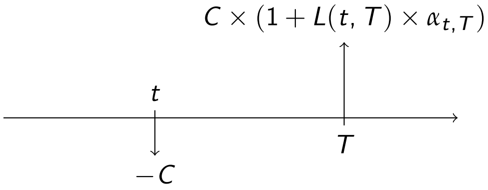
Tenor	EUR	USD	GBP	CHF	EURIBOR
O/N	-0.39029	0.3802	0.48313	-0.7596	-0.34
1W	-0.37643	0.401	0.48688	-0.7966	-0.357
2W					-0.352
1M	-0.34914	0.4352	0.50881	-0.7708	-0.343
2M	-0.30286	0.52425	0.54825	-0.747	-0.289
3M	-0.27286	0.63835	0.59088	-0.7284	-0.252
6M	-0.15571	0.9139	0.74625	-0.646	-0.144
9M					-0.078
1Y	-0.029	1.2441	1.02713	-0.5236	-0.013

Table 10: Source: Il Sole 24 Ore, April, 27 2016. All quotations are in percentage terms.

Computing Final Value using LIBOR rate

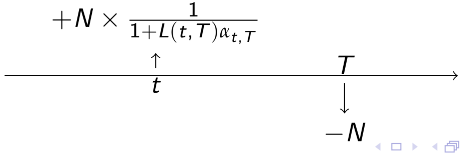
simple convention

$$FV_{t,T}(C) = C \times (1 + L(t,T) \times \alpha_{t,T}) \,. \tag{1}$$



Computing Present Value using LIBOR rates

$$PV_{t,T}(N) = N \times \frac{1}{1 + L(t,T)\alpha_{t,T}} \,. \tag{2}$$



Fact (LIBOR: Discounting and Compounding)

If $L(t, T)$ is the LIBOR rate quoted in t for expiry T, then:

- *the discount factor is*

$$P(t, T) = \frac{1}{1 + L(t, T) \times \alpha_{t,T}}$$

- *the final value is*

$$FV(t, T) = 1 \times (1 + L(t, T) \times \alpha_{t,T})$$

where

- *t is the value date (2 business days after the trade (fixing) date), and*
- *T is the anniversary date of the value date according to the rolling day rule modified following (end-of-month).*
- *$\alpha_{t,T}$ is the year fraction between t and T computed according to a particular day count convention, as in Table 9,*

The term structure of LIBOR rates

- The plot of $L(t, T)$ varying T is called the term structure of LIBOR rates.
- The plot of $P(t, T)$ varying T is called the term structure of money market discount factors.
- They can be plotted for maturities up to 1 year.

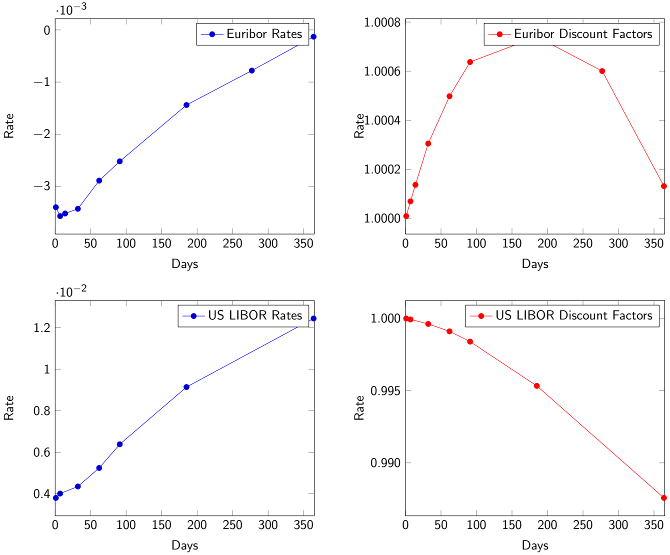


Figure 10: Top Left: Euribor Rates. **Top Right**: Euribor Discount Factors. **Bottom Left**: US LIBOR Rates. **Bottom Right**: US Discount Factors. Date: April 27th, 20016.

Example (Case Study. Building the discount curve in the money market)

See Excel File: BasicYields.xlsx Sheet: LIBOR

Let us consider EURIBOR quotes on the Fixing Date of April 27th, 2016 (W) (Value Date: April 29, 2016 (F)). Our steps: a) calculate the start and end dates for each of our quote; b) apply the modified following business day convention (yellow cells)

Notice that the Value Date is the last business day of the month, because 30-Apr-2016 is a Saturday. So the end of month (EOM) rule applies.

	Anniversary Date	D	M	Y	Adjusted Maturity	Modified Following
O/N	Th. 28 Apr. 2016	1			Th. 28 Apr. 2016	Th. 28 Apr. 2016
1W	F 6 May 2016	7			F 6 May 2016	F 6 May 2016
2W	F 13 May 2016	14			F 13 May 2016	F 13 May 2016
1M	T 31 May 2016 (EOM)		1		T 31 May 2016	T 31 May 2016
2M	Th 30 June 2016 (EOM)		2		Th 30 June 2016	Th 30 June 2016
3M	S 31 July 2016 (EOM)		3		**M 1 Aug 2016**	F 29 July 2016
6M	M 31 Oct. 2016 (EOM)		6		M 31 Oct. 2016	M 31 Oct. 2016
9M	Th 31 Jan. 2017 (EOM)		9		M Th Jan. 2017	Th 31 Jan. 2017
1Y	S 30 Apr. 2017 (EOM)		12		**Th 2 May 2017**	**F 28 Apr. 2017**

Table 11: Computing the Final Dates

Example (2. Building the term structure of discount factors)

Table 12: Computing Discount Factors. For example, the 1 year discount factor in the last row has been computed according to $\frac{1}{1+(-0.013\%)\times\frac{364}{360}} = \frac{1}{1+(-0.013\%)\times 1.01111} = 1.0001315$.

Contract	DAYS	$\alpha_{t,T}$	LIBOR	$P(t,T)$
O/N	1	0.002778	-0.0034	1.0000094
1 week	7	0.019444	-0.00357	1.0000694
2 weeks	14	0.038889	-0.00352	1.0001369
1 month	32	0.088889	-0.00343	1.0003050
2 months	62	0.172222	-0.00289	1.0004980
3 months	91	0.252778	-0.00252	1.0006374
6 months	185	0.513889	-0.00144	1.0007405
9 months	277	0.769444	-0.00078	1.0006005
12 months	364	1.011111	-0.00013	**1.0001315**

Example (3. Computing Final Value)

Given the Trade Date of April, 27, we borrow 1,000,000 EUR for 6 months tenor.

Trade	Wed. April 27th, 2016
Spot Lag	2 Bus Days
Value Date t	F Apr 29th, 2016 (EOM)
Maturity T	M Oct. 31, 2016
Days(t, T)	185
Act/360 $\alpha_{t,T}$	0.513888889
LIBOR	-0.1440%
Deposit in t	1,000,000.00
Final Value in T	999,260.00

1,000,000 × (1 + (-0.144%) × 185/360) = 999,260

The Final value is computed according to

$$FV_{t,T}(1,000,000) = 1,000,000 \times \left(1 - 0.144\% \times \frac{185}{360}\right) = 999,260.00.$$

Example (4b. Cash Flows Diagram)

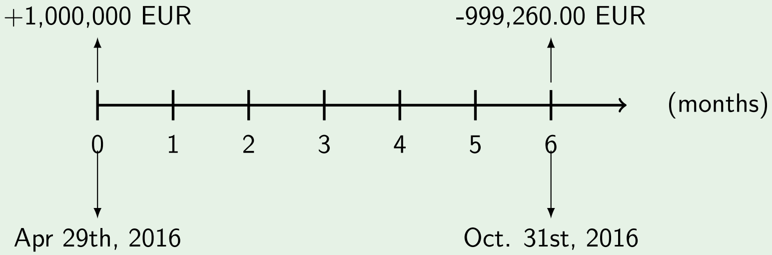


Figure 11: The cash flows in a EURIBOR deposit. The Final value is lower than the initial value due to the negative interest rate

Example (5. Computing Present value)

- Given the trade date of April, 27, we want to make a deposit today to receive 1,000,000 EUR in 6 months.
- The present value of 1,000,000 is

$$\begin{aligned} PV_{t,T}(1,000,000) &= \frac{1,000,000}{1 - 0.144\% \times 0.513888889} \\ &= 1,000,000 \times 1.000741 \\ &= 1,000,741 EUR. \end{aligned}$$

- This is the amount that we have to deposit today.
- In 6 months, we will receive 1,000,000 EUR.

Example (5b. Cash Flows Diagram)

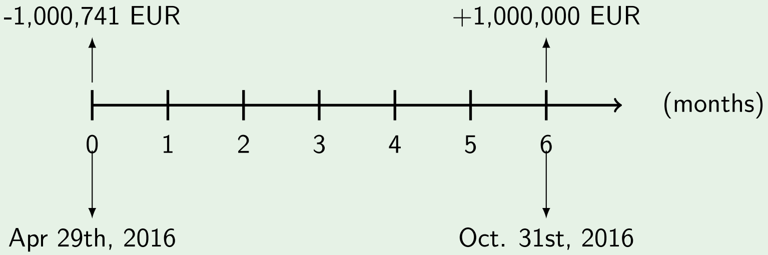


Figure 12: The cash flows in a EURIBOR deposit. The Final value is lower than the initial value due to the negative interest rate

Question

On April 27th, 2016, the US LIBOR rate for a 3 months tenor is 0.63835%. Compute:

- the Final Value of 1ml USD deposited on Apr 29th, 2016;
- the Present Value of 1ml USD to be paid on Jul 29th, 2016;

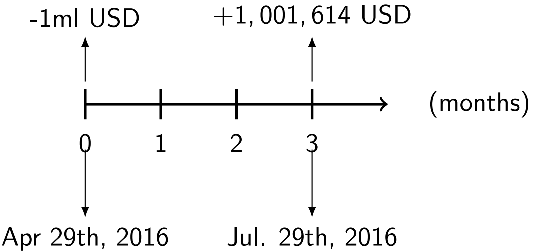
$$FV: \quad 1{,}000{,}000 \times \left(1 + 0.63835\% \times \frac{91}{360}\right)$$

Answer I

There are 91 days between the Start Date and the End Date. Therefore:

- **Final Value.** The Final Value of 1ml USD deposited on Apr 29th, 2016 is

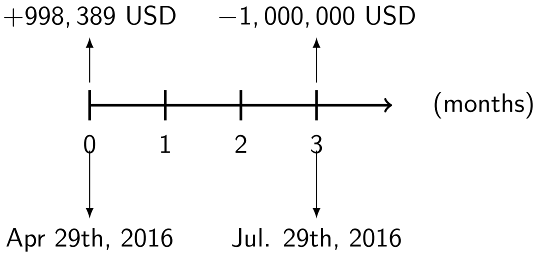
$$+1\text{ml} \times \left(1 + 0.63835\% \times \frac{91}{360}\right) = +1,001,614.$$



Answer II

- **Present Value.** The present value of 1ml USD to be paid on Jul 29th, 2016 is

$$\frac{+1\text{ml}}{1 + 0.63835\% \times \frac{91}{360}} = 998,389 USD$$



Example (Case Study: Using LIBOR Rate)

This case study is also presented in the Excel File BasicYields.xlsm, Sheet: Using LIBOR

1. Let us consider the following term structure of EURIBOR Rates on April 27, 20016.

Days	Accrual	Rate	DF
1	0.002778	-0.0034	1.000009
7	0.019444	-0.00357	1.000069
14	0.038889	-0.00352	1.000137
32	0.088889	-0.00343	1.000305
62	0.172222	-0.00289	1.000498
91	0.252778	-0.00252	1.000637
185	0.513889	-0.00144	1.000741
277	0.769444	-0.00078	1.000601
364	1.011111	-0.00013	1.000131

Table 13: Market Quotes

- We are interested in computing the present value of the following schedule of cash flows

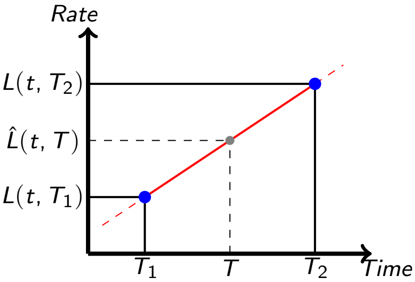
Time (Days)	Cash Flows
31	-100
79	300
289	12
350	-300
352	250
361	240

Table 14: Contract Cash Flows

- We observe that the cash flows do not fall exactly on market dates. So we have to proceed via some interpolation.
- The simplest procedure is to use linear interpolation on LIBOR rates.

- Let us suppose that we need the LIBOR rate for date T, i.e. $L(t, T)$:
    a. we have quotes for T_1 and T_2, $T_1 < T < T_2$, i.e. $L(t, T_1)$ and $L(t, T_2)$;
    b. Linear interpolation consists of

$$\hat{L}(t, T) = \frac{T_2 - T}{T_2 - T_1} \times L(t, T_1) + \frac{T - T_1}{T_2 - T_1} \times L(t, T_2). \tag{3}$$



d. Then the interpolated discount factor is

$$\hat{P}(t, T) = \frac{1}{1 + \hat{L}(t, T) \times \alpha_{t,T}}.$$

- So for example for the cash flow due in $T = 31$ days, the interpolated rate can be obtained using the $T_1 = 14$ days LIBOR rate and the $T_2 = 32$ one:

$$L(31days) = \frac{32-31}{32-14} \times (-0.352\%) + \frac{31-14}{32-14} \times (-0.343\%) = -0.3435\%.$$

- Then we can compute the corresponding discount factor

$$P(t, t+31\text{days}) = \frac{1}{1 - 0.3435\% \times \frac{31}{360}} = 1.000296.$$

- Therefore, the present value of the due cash flow is

$$PV(-100) = -100 \times 1.000296 = -100.03.$$

- We can proceed in a similar way for all relevant dates. We obtain the following Table

Time (Days)	Cash Flows	Interp. LIBOR	Interp. DF	PV(CF)
31	-100	-0.00344	1.000296	-100.03
79	300	-0.00267	1.000587	300.18
289	12	-0.00069	1.000555	12.01
350	-300	-0.00023	1.000228	-300.07
352	250	-0.00022	1.000215	250.05
361	240	-0.00015	1.000153	240.04
			Sum	402.1751

Table 15: Computing the present value of the cash flows

- The fair value of the cash flows schedule is 402.1751 EUR.

Question

- The 32 days US LIBOR rate is 0.004352.
- The 62 days US LIBOR rate is 0.005242.
- Determine the Present Value of 100 USD due in 50 days.
- If I invest today 200 USD for 50 days, how much do I get at the End date?

Answer

- The interpolated 50 days US LIBOR rate is

$$\begin{aligned} & \frac{62-50}{62-32} \times 0.004352 + \frac{50-32}{62-32} \times 0.005242 \\ = \quad & 0.4 \times 0.004352 + 0.6 \times 0.005242 \\ = \quad & 0.0048863. \end{aligned}$$

- The interpolated discount factor is

$$\frac{1}{1 + 0.0048863 \times \frac{50}{360}} = 0.999322.$$

- The Present Value of 100 USD due in 50 days is

$$100 \times 0.999322 = 99.9322.$$

- If I invest today 200 USD for 50 days, at the End Date I have

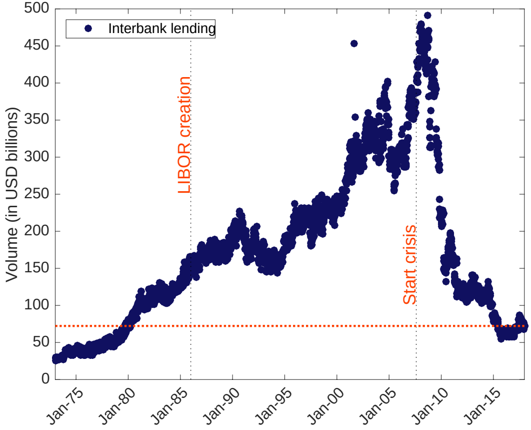
$$200 \times \left(1 + 0.0048863 \times \frac{50}{360}\right) = 200.1357306.$$

LIBOR fallback

- Fallback refers to the replacement of LIBOR with new indexes.
- The idea of the new index is that it should reflect real transactions rather than expectations.
- In part, this has already happened for US Fed Fund (SOFR, Secured Overnight Financing Rate). SOFR is a broad measure of the cost of borrowing cash overnight collateralized by Treasury securities.
- Euribor should be replaced in the future by ESTER (Euro Short Term Rate), that will be the European Risk Free Rate (RFR).
- ESTER will replace EONIA (and EURIBOR) as the most important interest rate in Europe.
- Useful readings:
  https://www.clarusft.com/libor-fallbacks/
  https://www.ecb.europa.eu/paym/initiatives/interest_rate_benchmarks/WG_euro_risk-free_rates/html/index.en.html

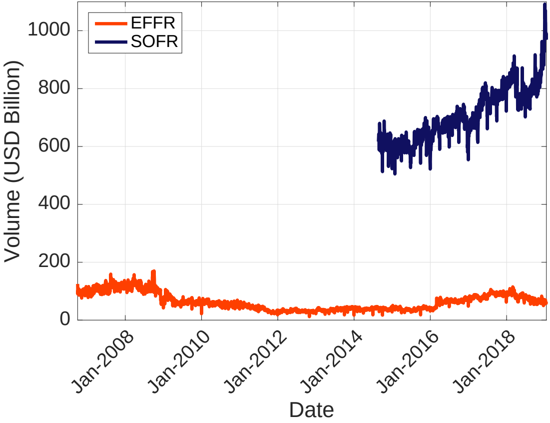
LIBOR – decrease of importance

Unsecured lending has decreased a lot since the crisis.



Source: US Federal Reserve – All Commercial Banks in the United States – Interbank loans

Unsecured versus secured lending



Effective Federal Fund Rate (unsecured) and Secured Overnight Financing Rate. Data from the Federal Reserve Bank of New York.

New benchmarks

There is a push by regulators, e.g. FSB, to move away from IBOR and to RFR.

- GBP (SONIA): Published by Bank of England. Unsecured.
- CHF (SARON): Switch from TOIS to SARON end 2017. Secured.
- USD (SOFR): Published by the Fed since April 2018. Secured.
- EUR (ESTER): To be published by ECB from October 2019. Unsecured.

Zero-Coupon Bond

Zero-Coupon Bond

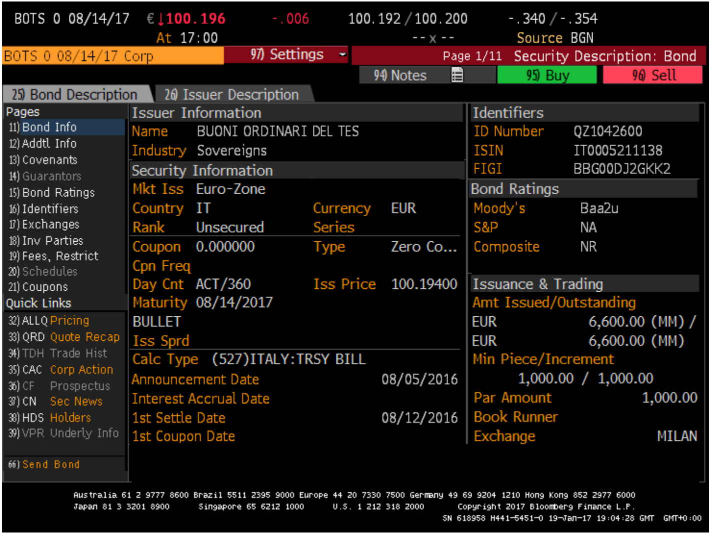


Figure 13: A Zero-Coupon bond

Zero Coupon Bond: Cash Flows Diagram

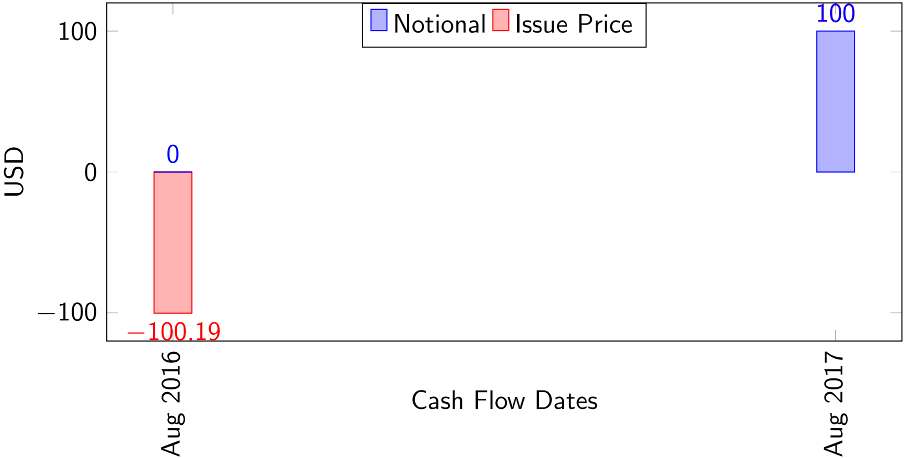


Figure 14: Cash Flows of a zero-coupon Bond

Main Terms

- **Announcement Date**: the first day the public will receive information regarding the new security issue.
- **Issue size**: number of bonds issued multiplied by the face value.
- **Issue Price**: the price paid by the buyer at the issue date.
- **Principal (Par Amount, Maturity Value, Notional, Face Value)**: the amount of money the issuer will pay the holder of a bond at the maturity date.
- **Maturity**: the date at which the principal (notional, face value) is redeemed, i.e. the debt will cease to exist.

The Bond Schedule

Cash Flows of a Zero-Coupon Bond		
Issue Date	August 12th, 2016	
Issue Price	100.194	
Maturity Date	August 14th, 2017	
Face Value	100	
Coupon	0	(zero-coupon bond)
Day Count Convention	ACT/360	(for yield computation)

Dates	Adj. Dates T_i	Cash Flows
Fr Aug 12th, 2016	Fr Aug 12th, 2011	-100.194
Sat Aug 12th, 2017	Mon Aug 14th, 2017	100

Coupon Bond

Coupon Bond I

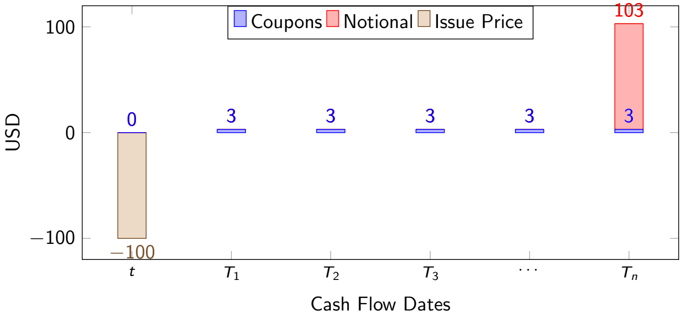


Figure 15: Cash Flows of a Coupon-Bearing Bond

- Fixed payments (coupon) at predeterminated dates.
- Repayment of the principal (nominal value) at maturity (usually greater than 3 years at issue).

Coupon Bond II

- Examples are
  - Treasury Notes (mature in two to ten years) and Treasury Bonds (have the longest maturity, from twenty years to thirty years) in the US;
  - Bunds in Germany;
  - BTPs in Italy;
  - Gilts in UK;
  - OATs in France;
  - Bonos in Spain.
- Issuers can be governments, financial and corporate firms.

Coupon Bond

Figure 16: Fixed rate bond

Main Terms

- **Interest Accrual Date**: date at which the first coupon starts to accrue.
- 1st Settle Date the date from which a bondholder is entitled to receive interest payment (coupons).
- **1st Coupon Date**: date at which the first coupon will be paid.
- **Coupon**: the interest amount that is paid periodically (e.g. 4.5% per year) to the bondholder by the issuer up to the maturity date; It can be fixed or floating.
- **Period or Frequency**: here Annual, i.e. the number of coupons received in a given year.
- **Calc Type**: the method used to determine the yield of the bond, i.e. the compounding method.
- **Day Count Convention**: ACT/ACT. The choice of time measurement used in computing coupons.

Rolls Dates

- Each bond has interest payments on roll dates that are the same each year (aside adjustments for holidays and weekends).
- For example, if a bond has an issue date of 1-Apr-2016 and matures on 1-Apr-2020, and has semi-annual coupons, the normal roll dates would be 1st April and 1st October each year. Each semester we receive half (notional) coupon.

Example (Regular Roll Dates)

Issue Date	April 1st, 2016
Maturity Date	April 1st, 2020
Frequency	2
Coupon	5

Times	A. 1st-16	O. 1st- 16	A. 1st-17	$\cdots$	$\cdots$	A. 1st-20
Days		183	182	$\cdots$	$\cdots$	182
Cash Flow	Issue Price	2.5	2.5	$\cdots$	$\cdots$	102.5

- We can have bonds with irregular roll dates, e.g. a so called short first or last coupon.

The Bond Schedule

The coupon payment at date T_i is equal to

$$C(T_i) = c \times \alpha_{i-1,i} \times N,$$

where

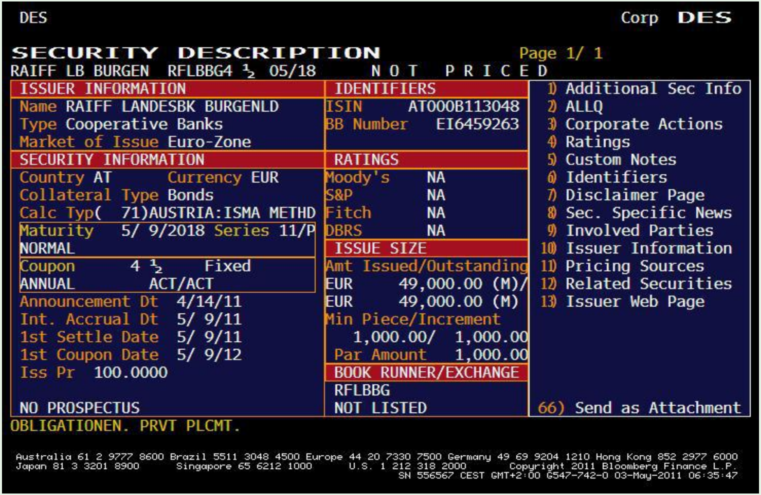
- c: yearly coupon and N is the face value;
- $\alpha_{i-1,i}$: is called coupon tenor, i.e. the fraction of time between coupon dates T_{i-1} and T_i;
- for regular coupon periods the convention ACT/ACT ISMA means

$$\alpha_{i-1,i} = \frac{1}{Frequency},$$

e.g. if the coupons are semi-annual, then $Frequency = 2$ and so

$$\alpha_{i-1,i} = \frac{1}{2}.$$

Example (Case Study: Payment schedule of a Coupon Bond)



Example (1. Main Informations)

Main Information	
ISIN	AT000B113048
Issue Date	April 14th, 2011
Maturity Date	May 9th, 2018
Frequency	1
Coupon	4.5
First Accrual Date	May 9th, 2011
Day Count Convention	ACT/ACT

Here, there is no odd coupon because the coupon dates are the same each year (aside adjustements for holidays and weekends).

Example (2. Coupond Dates and cash flows)

1. The periodic coupon is equal to

$$C(T_i) = c \times \alpha_{i-1,i} \times N = 4.5\% \times 1 \times 100 = 4.5,$$

2. Unadjusted coupon dates are on May 9th of each year, starting in 2011 (First Accrual Date) and ending in 2018 (Maturity Date).
3. We can complete the first column of the Table below. The first letter refers to the day of the week.
4. Then we adjust the payment dates for weekends. We should adjust for holidays as well (but I do not have them for Austria!).

Example (3. Cash Flows Table)

We have the payment schedule below

Coupon Dates	Adj. Cpn Date T_i	$\alpha_{i-1,i}$	Coupon	Notional	Cash Flows
m may 9, 2011	m 9 may, 2011				
w 9 may, 2012	w 9 may, 2012	1	4.5	0	4.5
th 9 may, 2013	th 9 may, 2013	1	4.5	0	4.5
f 9 may, 2014	f 9 may, 2014	1	4.5	0	4.5
s 9 may, 2015	m 11 may, 2015	1	4.5	0	4.5
m 9 may, 2016	m may 9, 2016	1	4.5	0	4.5
tu 9 may, 2017	tu 9 may, 2017	1	4.5	0	4.5
w 9 may, 2018	w 9 may, 2018	1	4.5	100	104.5

Example (4. Day Count Convention)

6. Then in the third column we compute the coupon accrual factor according to the day count convention ACT/ACT (ISMA Method). Accordingly, we have

$$\text{Accrual Factor} = \frac{\text{Days in the period}}{\text{Denominator}}$$

   where

   - **Days in the period**: actual number of days from and including the last coupon date to, but excluding, the current value date.
   - **Denominator**: is the actual number of days in the coupon period multiplied by the number of coupon periods in the year.

7. In practice, having an annual frequency, according to the ACT/ACT convention, this accrual factor is always equal to 1.

Coupon Bond with semi-annual coupons

Figure 17: Fixed rate bond

Cash flows of a bond with semi-annual coupons

- In this case the bond has semi-annual coupons.
- Therefore every semester, the bond pays a coupon equal to

$$C(T_i) = 2.125\% \times \frac{1}{2} \times 100 = 1.0625.$$

- Notice that US treasury securities follow *month-end* convention, so if the maturity is at month-end, then the assumed coupon dates are also at month-end (e.g. March 31st rather March 30st)
- The dates at which these coupons are paid are the 31st of May and the 30th of November of each year, up to Nov. 30th, 2023.
- The first coupon is paid on May 31st, 2017.
- Last coupon is paid on November 30st, 2023.
- On this date we also receive back the principal value.

Question.

Consider a bond with quarterly coupons, coupon at 4%, face value of 100, expiry in 12 months. Build the bond payment schedule.

Answer.

We have

Time	Coupon Tenor	Cpn	Notional	Cash Flow
0				
0.25	0.25	1	0	1
0.5	0.25	1	0	1
0.75	0.25	1	0	1
1	0.25	1	100	101

Question.

Consider a bond with quarterly coupons, coupon at 4%, face value of 100, expiry in 8 months. Build the bond payment schedule.

Answer.

We have

Time	Coupon Tenor	Cpn	Notional	Cash Flow
0				
2/12	0.25	1	0	1
5/12	0.25	1	0	1
8/12	0.25	1	100	101

Table 16: In order to build the payment schedule, (normally) we start from the bond maturity and in a backward procedure we determine all the previous payment dates.

Market Quotations Coupon Bond I

- When you buy a bond in the secondary market, in general this will happen between coupon dates.
- This means that you have to pay the seller a full price, i.e. to recognize to him the matured coupon fraction.
- The full price (or dirty price or invoice price or gross price), consists of two components:
  - **Clean Price**:
    Quoted price of the bond without the accrued interest.
  - **Accrued interest**:
    This amount compensates the seller of the bond for the coupon interest earned from the time of the last coupon payment to the settlement date of the bond.
  - PV **Invoice Price**= Clean Price+Accrued interest.
  - A Bond quotes **at Par** if **Invoice Price=Face Value**.
  - A Bond quotes **above the Par** if **Invoice Price>Face Value**.
  - A Bond quotes **below the Par** if **Invoice Price<Face Value**.

Market Quotations Coupon Bond II

Formula for Accrued Interest

- In the majority of cases the accrued interest which has to be added to the price is equal to:

$$AI = \frac{c}{m} \times \frac{d_1}{d_2},$$

where

c is the annual coupon rate,

m the coupon frequency,

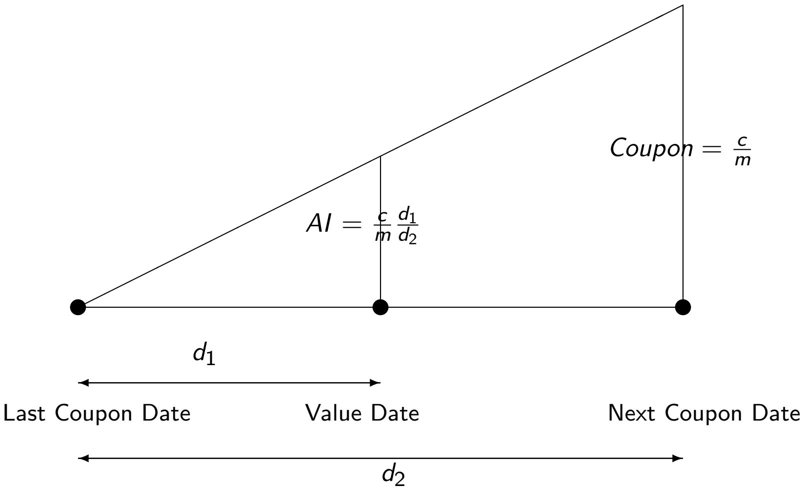
d_1 is the number of days accrued since last coupon payment,

d_2 is the number of days in the coupon period.

- Therefore

$$\textit{Gross Price}=\textit{Clean Price}+\textit{Accrued Interest}.$$

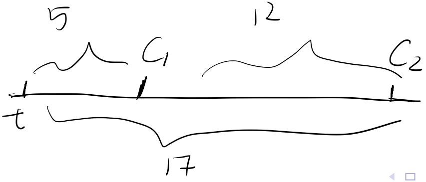
Computation of the Accrued Interest



Consider a German Bund with annual coupons, coupon at 2%, face value of 100, expiry in 17 months. The bond clean price is 103. Build the bond cash flows schedule. For your convenience measure time in months and round to the second digit.

$FV = 100$ $\quad$ $coupon = 2\%$ $\quad$ $T - t = \frac{17}{12}$

$Clean\ Price = 103$ $\quad$ $100 \times 2\% = 2$



$2 \times \frac{7}{12} = \frac{14}{12}$

$= 1.1667$

$Gross\ Price = 104.1667$

Answer.

Time (months)	CF	Amount
0	-Dirty Price	-104.17
5	Cpn	2
17	Cpn+FV	102

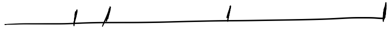
Table 17: The coupon period is 12 months, i.e. $m = 1$. The periodic coupon is 2%. Last coupon has been paid 7 months ago. The Accrued Interest is computed according to $2\% \times \frac{7}{12} \times 100 = 1.17$. Therefore the Dirty Price is $103 + 1.17 = 104.17$.

Question.

Consider a US Treasury Note with semi-annual coupons, coupon at 2%, face value of 100, expiry in 13 months and clean price of 98. Build the bond cash flows schedule. For your convenience measure time in months and round to the second digit.

$100 \times 2\% \times 0.5 = 1$

$1 \times \frac{5}{6} = 0.8333$



Answer.

Time (months)	CF	Amount
0	-Dirty Price	$-98 - 0.83 = -98.83$
1	Cpn	$2\% \times \frac{1}{2} \times 100 = 1$
7	Cpn	$2\% \times \frac{1}{2} \times 100 = 1$
13	Cpn+FV	$2\% \times \frac{1}{2} \times 100 + 100 = 101$

Table 18: The coupon period is semi-annual, i.e. $m = 2$. The semi-annual coupon is $2\% \times \frac{1}{2} \times 100 = 1$. Last coupon has been paid 5 months ago. Accrued Interest is computed according to $2\% \times \frac{1}{2} \times \frac{5}{6} \times 100 = 0.83$. Therefore the Dirty Price is $98 + 0.83 = 98.83$.

Counting Accrued Days I

- There are several methods of counting the number of days of accrued interest, although they are subject to variation
  - Actual calendar days, including 29 February if it occurs in the period;
  - 30-day European Method, i.e.:
    - If $D_1 = 31$ set it to 30;
    - If $D_2 = 31$ set it to 30;
    - Number of days accrued

$$(D_2 - D_1) + 30 \times (M_2 - M_1) + 360 \times (Y_2 - Y_1).$$

  - 30-day U.S. Method, i.e.
    - If $D_1 = 31$ set it to 30;
    - If $D_2 = 31$, and $D_1 = 30 or 31$ set $D_2 = 30$, otherwise leave as 31
    - Number of days accrued

$$(D_2 - D_1) + 30 \times (M_2 - M_1) + 360 \times (Y_2 - Y_1).$$

    and a coupon falling on last day of February it is treated as if it were on the 30th of February.

Counting Accrued Days II

- For US Treasury coupon securities and Italian BTPs and CCTs, the convention is to use the actual number of days since the last coupon date and the actual number of days between coupon payments, i.e. an actual/actual ratio.
- For US corporate, municipal, and agencies, the day count convention is 30/360: Each month is assumed to have 30 days and each year is assumed to have 360 days.
- For international securities, ISMA rule 251 specifies how accrued interest are calculated for fixed and floating securities. All securities issued after 31 December 1998 use actual calendar days.
- In general, it is also specified the method of rounding accrued interests, e.g. 4 digits for US Treasuries and 5 digits for Italian BTPs.

Counting Accrued Days III

Example ▸

Consider a bond which follows the U.S. method with semi-annual coupon payments maturing on 31 August 2005. The other coupon payment is normally the last day of February. The number of days accrued interest is on the following dates.

	Days accrued		*Days accrued*
27 February 1996	*177*	*27 February 1997*	*177*
28 February 1996	*178*	*28 February 1997*	*0*
29 February 1996	*0*		
1 March 1996	*1*	*1 March 1997*	*1*
30 August 1996	*180*	*30 August 1997*	*180*
31 August 1996	*0*	*31 August 1997*	*0*

The following figure shows a comparison of some of the differences between the ISMA/European, the U.S. and the actual methods of calculating the number of days accrued.

		No. of days between Date 1 & Date 2		
Date 1	*Date 2*	*ISMA Method*	*U.S. Method*	*Actual*
29 July	*31 August*	*31*	*32*	*33*
30 July	*31 August*	*30*	*30*	*32*
31 July	*31 August*	*30*	*30*	*31*
1 August	*31 August*	*29*	*30*	*30*
29 July	*1 September*	*32*	*32*	*34*
30 July	*1 September*	*31*	*31*	*33*
31 July	*1 September*	*31*	*31*	*32*
1 August	*1 September*	*30*	*30*	*31*

Figure 18: Source: Bond Markets. Structures and Yield Calculation. by Patrick Brown, pages 8-9

Example (Accrued Interest for BTP 15.01.2008 (IT0003413892))

Table 19: The Issuer considers all periods equal among each other (30/360). For example, for fixed-rate securities issued on 1 January, with an annual coupon of 4%, the Issuer will pay a coupon of 2% each semester, regardless of the actual duration of the semester. The accrued interest is rounded to the 6th decimal per 1000 euros of capital, for the issuance of Government securities through ordinary auctions and in exchange transactions. On the secondary market, however, the convention of 5 decimals per 100 euros of capital applies. Therefore the accrued interest payable will be 1.02717 per 100 euros.)

Accrued interest for BTP		
Trade Date		Tues Oct. 28th, 2003
Value date (3 Bus. Days)		Oct. 31st, 2003
Last coupon date		July 15th, 2003
Next coupon date:		Jan. 1st, 2004.
days(15/07/2003; 31/10/2003)	d_1	108
days(15/07/2003; 15/01/2004)	d_2	184
Coupon Rate	c	3.5
Coupons in the year	m	2
Accrued Interest	$\frac{c}{m} \times \frac{d_1}{d_2}$	$\left(\frac{4}{2}\right) \times \frac{108}{184} = 1.1739$

Example (Accrued Interests for US Bonds and Municipals)

- Consider a T-note whose last coupon payment was on March 1 and its next coupon is six months later on September 1.
- Suppose the bond is purchased with a settlement date of July 17.
  - The actual number of days between coupons is: July 17-July 31 = 14 days; August = 31 days; September 1 = 1 days. **Total = 46 days**.
  - The actual number of days in the **coupon period** (sometimes referred to as the basis) is **184 days**.
- If the previous bond were a municipal or a corporate (and the day count convention is 30/360), then
  - the days between coupon would be: Remainder of July = 13 days; August = 30 days; September 1 = 1 day. **Total = 44 days**.
  - The number of days in the **coupon period** would be **180**.

Market Quotations Coupon Bond

Bond Yields

TREASURY ISSUES

Tuesday, January 31, 2017

Prices and yields for on-the-run Treasurys, or the most recently issued U.S. Treasury securities, for various maturities. Data as of 3 p.m. ET.

Maturity	Coupon	Current price	Previous price	Change	Yield
02/23/17	...	99.97	99.97	0.002	0.472
05/04/17	...	99.87	99.87	unch.	0.518
08/03/17	...	99.68	99.68	unch.	0.636
01/04/18	...	99.30	99.28	0.016	0.764
01/31/19	**1.125**	99.84	99.83	0.016	1.204
01/15/20	**1.375**	99.75	99.72	0.031	1.462
01/31/22	**1.875**	99.84	99.70	0.148	1.908
01/31/24	**2.250**	100.02	99.80	0.219	2.248
11/15/26	**2.000**	96.09	95.80	0.297	2.451

Figure 19: Quotes of US Bonds. Source: Wall Street Journal http://online.wsj.com/mdc/public/page/2_3020-treasury.html

Example

1. On Tuesday, Jan. 31, 2017 we buy the Nov. 15, 2026 US Bond.
2. For US bonds, there is 1 business day lag. So the Value Date is Feb, 1st, 2017
3. The Clean Price is 96.09.
4. The coupon rate on annual basis is 2%.
5. Coupon dates are on Nov. 15 and May 15, i.e. coupons are paid every six months.
6. Last coupon has been paid on Nov. 15, 2016.
7. Next coupon will be paid on May. 15, 2016.
8. There are 181 actual days in the coupon period.
9. 78 actual days are elapsed since last coupon payment.
10. The Accrued Interest is
$$\frac{2\%}{2} \times \frac{78}{181} \times 100 = 0.4309.$$
11. The Gross Price to be paid on Feb 1st is
$$96.09 + 0.4309 = 96.5209.$$

Example (Case Study: Building the Payment Schedule of an US Bond)

Excel file: FI_BasicYields.xlsm
Sheet: BondSchedule

	A	B	C	D	E	F
1	**US BOND: Bond Schedule**					
2	**US BOND**					
3	ISIN	912828U24				
4	**BOND INFORMATION**					
5	Face Value	100				
6	Annual coupon	2	D	M	Y	
7	Maturity	15 November 2026	16	11	2026	
8	Basis	1				
9	Number of coupons in the year	2				
10	**MARKET INFORMATION**					
11	Trade Date	02 February 2017				
12	Clean Price	95.00000				
13	**BOND CALCULATOR**		D	M	Y	
14	Value Date	03 February 2017	3	2	2017	=WORKDAY(B11,1,)
15	Payment of last coupon	15 November 2016				=COUPPCD(ValueDate,B7,B9,B8)
16	Date of next coupon	15 May 2017				=COUPNCD(ValueDate,B7,B9,B8)
17	Number of remaining coupons	20	n			=COUPNUM(ValueDate,B7,B9,B8)
18	Days since last coupon	80	u			=COUPDAYBS(ValueDate,B7,B9,B8)
19	Days to next coupon	181	v			=COUPDAYS(ValueDate,B7,B9,B8)
20	Accrued Interest	0.44199				=ROUND((B6/2)*B18/(B19),5)
21	Market Gross Price	95.44199				=B12+B20

Figure 20: Bond Information

Example (Case Study: (...continued))

Table 20: Payment Dates and Cash Flows

Cpn Nr	Coupon Starts	Payment Ends	Adj. Starting Date	Adj. Ending Days	Tenor	TTM	AF	CF
1	Tu-Nov 15-2016	M-May 15-2017	Tu-Nov 15-2016	M-May 15-2017	181	101	0.5	1
2	M-May 15-2017	W-Nov 15-2017	M-May 15-2017	W-Nov 15-2017	184	285	0.5	1
3	W-Nov 15-2017	Tu-May 15-2018	W-Nov 15-2017	Tu-May 15-2018	181	466	0.5	1
4	Tu-May 15-2018	Th-Nov 15-2018	Tu-May 15-2018	Th-Nov 15-2018	184	650	0.5	1
5	Th-Nov 15-2018	W-May 15-2019	Th-Nov 15-2018	W-May 15-2019	181	831	0.5	1
6	W-May 15-2019	F-Nov 15-2019	W-May 15-2019	F-Nov 15-2019	184	1015	0.5	1
7	F-Nov 15-2019	F-May 15-2020	F-Nov 15-2019	F-May 15-2020	182	1197	0.5	1
8	F-May 15-2020	Su-Nov 15-2020	F-May 15-2020	F-Nov 13-2020	182	1379	0.5	1
9	Su-Nov 15-2020	Sa-May 15-2021	F-Nov 13-2020	F-May 14-2021	182	1561	0.5	1
10	Sa-May 15-2021	M-Nov 15-2021	F-May 14-2021	M-Nov 15-2021	185	1746	0.5	1
11	M-Nov 15-2021	Su-May 15-2022	M-Nov 15-2021	F-May 13-2022	179	1925	0.5	1
12	Su-May 15-2022	Tu-Nov 15-2022	F-May 13-2022	Tu-Nov 15-2022	186	2111	0.5	1
13	Tu-Nov 15-2022	M-May 15-2023	Tu-Nov 15-2022	M-May 15-2023	181	2292	0.5	1
14	M-May 15-2023	W-Nov 15-2023	M-May 15-2023	W-Nov 15-2023	184	2476	0.5	1
15	W-Nov 15-2023	W-May 15-2024	W-Nov 15-2023	W-May 15-2024	182	2658	0.5	1
16	W-May 15-2024	F-Nov 15-2024	W-May 15-2024	F-Nov 15-2024	184	2842	0.5	1
17	F-Nov 15-2024	Th-May 15-2025	F-Nov 15-2024	Th-May 15-2025	181	3023	0.5	1
18	Th-May 15-2025	Sa-Nov 15-2025	Th-May 15-2025	F-Nov 14-2025	183	3206	0.5	1
19	Sa-Nov 15-2025	F-May 15-2026	F-Nov 14-2025	F-May 15-2026	182	3388	0.5	1
20	F-May 15-2026	Su-Nov 15-2026	F-May 15-2026	F-Nov 13-2026	182	3570	0.5	101

Example (Case Study: (...continued))

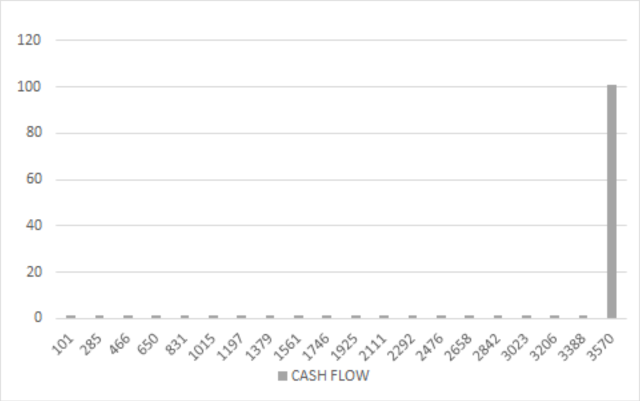


Figure 21: Bond Cash Flows

Pricing a Coupon Bond

Fact (Pricing a Coupon Bond)

- *The coupon bond can be seen as a strip of zero-coupon bonds, so by the law of unique price, the bond price (invoice price) is the sum of discounted values cash flows*

$$B\left(c, t; t_1, \ldots, t_n\right) = \sum_{i=1}^{n} c\alpha_{t_{i-1},t_i} P\left(t, t_i\right) + 1 \times P\left(t, t_n\right). \qquad (4)$$

  where

  c is the coupon (on an annual basis);

  n is the number of remaining coupons;

  t_i is the payment date of the i-th coupon;

  α_{t_{i-1},t_i} is the length (in years, according to a given day count convention) of the i-th coupon period.

- *The above theoretical price is an estimate of the* ***gross (invoice) price****.*
- *If you are interested in the clean price you have to subtract to the above price the* ***accrued interest****.*

- This evaluation requires the knowledge of the term structure of discount

Example (1. Pricing the coupon bond relative to zeroes)

Let the price of three zero-coupon bonds with maturities of 1-3 years

$$P(0,1) = 0.9346, P(0,2) = 0.8573, P(0,3) = 0.7722.$$

The equilibrium price B of a 3-year, 8% annual coupon bond with face value of 100 is 97.73:

$$B = (0.9346 + 0.8573 + 0.7722) \times 8 + 0.7722 \times 100 = 97.7316.$$

Example (2. Building an arbitrage)

If the 8% bond trades at 95, we can do the following arbitrage:

- Buy the bond for 95.
- Consider three stripped zero coupons and sell them:

  1-year zero with Face Value of 8: Selling Price $= 8 \times 0.9346 = 7.4766$

  2-year zero with Face Value of 8: Selling Price $= 8 \times 0.8573 = 6.8587$

  3-year zero with Face Value of 108: Selling Price $= 108 \times 0.7722 = 83.3958$
- Sale of strip bonds $= 97.73$.
- Risk-free profit $= 97.73 - 95 = 2.73$.
- Given this risk-free opportunity, arbitrageurs would implement this strategy of buying and stripping the bond until the price of the coupon bond was bid up to equal its equilibrium price of 97.73.
- At that price, the arbitrage would disappear.
- Viceversa, if the bond trades above 97.73.

Recovering Discount factors from bond prices

Bootstrapping

- An important procedure in the marketplace is to recover the term structure of discount factors from bond having different maturities.
- In a simplified setup, with bonds expiring at equally space dates, this consists in solving forward a linear system of equations.
- In general, the procedure is more involved.
- This will be discussed in the lecture on Bootstrapping.

Par Coupon Rate

- Consider the bond pricing formula

$$B\left(c, t; t_1, \dots, t_n\right) = \sum_{i=1}^{n} c\alpha_{t_{i-1},t_i} P\left(t, t_i\right) + 1 \times P\left(t, t_n\right). \quad (5)$$

- Choose the coupon rate c so that $B = 1$, i.e.

$$1 = \sum_{i=1}^{n} c\alpha_{t_{i-1},t_i} P\left(t, t_i\right) + 1 \times P\left(t, t_n\right). \quad (6)$$

- We have

$$c_n = \frac{1 - P\left(t, t_n\right)}{\sum_{i=1}^{n} \alpha_{t_{i-1},t_i} P\left(t, t_i\right)}. \quad (7)$$

This coupon value is called **par coupon rate** or **swap rate**.

Example (Building the Par Coupon Curve)

- Let us suppose we have the term structure of annually compounded spot rates as in Table below.

Term	1	2	3	4	5
Spot Rate	2.3840%	2.8700%	3.2300%	3.5200%	3.7500%

  We aim to build the par-coupon curve.

- **Step 1**: Determine Discount factors according to the formula $P(t,T) = (1 + Y(t,T))^{-(T-t)}$.

Term	1	2	3	4	5
Spot Rate	2.3840%	2.8700%	3.2300%	3.5200%	3.7500%
DF	0.9767	0.9450	0.9090	0.8708	0.8319

Example

- **Step 3**: For each date, compute the Annuity $A_n = \sum_{i=1}^{n} \alpha_{i-1,i} P(t, T_i)$

Term	1	2	3	4	5
Spot Rate	2.3840%	2.8700%	3.2300%	3.5200%	3.7500%
Tenor	1	1	1	1	1
DF	0.9767	0.9450	0.9090	0.8708	0.8319
Annuity	0.9767	1.9217	2.8307	3.7015	4.5334

- **Step 4** Apply formula 7

Term	1	2	3	4	5
Spot Rate	2.3840%	2.8700%	3.2300%	3.5200%	3.7500%
Tenor	1	1	1	1	1
DF	0.9767	0.9450	0.9090	0.8708	0.8319
Annuity	0.9767	1.9217	2.8307	3.7015	4.5334
Par Yield	2.3840%	2.8631%	3.2134%	3.4913%	3.7085%

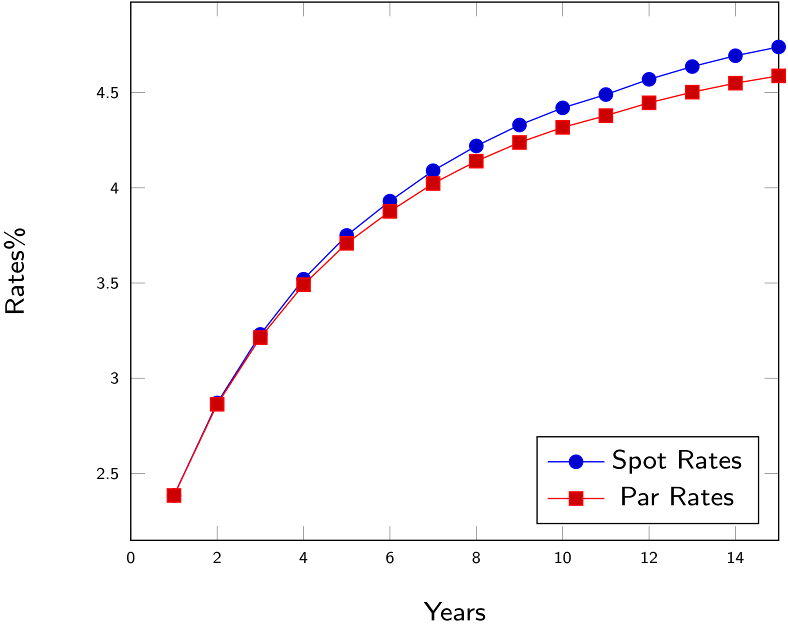


Figure 22: Term structures of spot and par rates.

From Par Rates to Discount Factors

- Consider the formula 7 and take $n = 1$, i.e. $c_1 = (1 - P(t, T_1))/(1 + \alpha_{T_0,T_1} P(t, T_1))$.
- Solve for $P(t, T_1)$. We have

$$P(t, T_1) = \frac{1}{1 + c_1 \alpha_{T_0,T_1}}.$$

- Take $n = 2$, i.e. $c_2 = \frac{1 - P(t,T_2)}{1 + \alpha_{T_0,T_1} P(t,T_1) + \alpha_{T_1,T_2} P(t,T_2)}$, and solve for $P(t, T_2)$:

$$P(t, T_2) = \frac{1 - c_2 \alpha_{T_0,T_1} P(t, T_1)}{1 + c_2 \alpha_{T_1,T_2}}.$$

- In general, we have

$$P(t, T_n) = \frac{1 - c_n \sum_{i=1}^{n-1} \alpha_{T_{i-1},T_i} P(t, T_i)}{1 + c_n \alpha_{T_{n-1},T_n}}. \tag{8}$$

i.e. given the term structure of par rates we can recover the term structure of discount factors.

Example

- **Step 1**: Get $P(0,1)$:

Years	1	2	3	4	5
Par	2%	2.20%	2.40%	2.50%	2.70%
Tenor	1	1	1	1	1
DF	98.04%				
Annuity	98.04%				

where

$$P(0,1) = 98.04\% = \frac{1}{1 + 2\% \cdot 1}$$

and

$$A(1) = 1 \cdot 98.04\%.$$

Example

- **Step 2** Get $P(0,2)$:

Years	1	2	3	4	5
Par	2%	2.20%	2.40%	2.50%	2.70%
Tenor	1	1	1	1	1
DF	98.04%	95.74%			
Annuity	98.04%	193.78%			

where

$$P(0,2) = 95.74\% = \frac{1 - 0.9804}{1 + 2.2\% \cdot 1}$$

and

$$A(2) = A(1) + 1 \cdot 95.74\% = 1.9378.$$

Example

- **Step 3** Get $P(0,3)$:

Years	1	2	3	4	5
Par	2%	2.20%	2.40%	2.50%	2.70%
Tenor	1	1	1	1	1
DF	98.04%	95.74%	93.11%		
Annuity	98.04%	193.78%	286.89%		

where

$$P(0,3) = 93.11\% = \frac{1 - 2.40\% \cdot 1.9378}{1 + 2.4\% \cdot 1}$$

and

$$A(3) = A(2) + 1 \cdot 93.11\% = 2.8689.$$

Example

- **Step 4** Get $P(0,4) - P(0,5)$:

Years	1	2	3	4	5
Par	2%	2.20%	2.40%	2.50%	2.70%
Tenor	1	1	1	1	1
DF	98.04%	95.74%	93.11%	90.56%	87.45%
Annuity	98.04%	193.78%	286.89%	377.45%	464.90%

Yield to maturity

- It is the rate that equates the price of the bond, B, to the PV of the bond's cash flow (CF); it is the internal rate of return, IRR, of a bond. It can be computed according to different conventions.
- In the US market, it is convention to use semi-annual compounding and measuring time in semesters.
- More details in the Appendix at page 171.

YTM

- It is the rate that equates the price of the bond, B, to the PV of the bond's cash flow (CF); it is the internal rate of return, IRR, of a bond. It can be computed according to different conventions.
- The intersection point between the horizontal line (at the level given by the bond market price) and the curved line (representing the theoretical coupon bond price for different values of y) gives the yield to maturity (in figure approximately equal to 6.24%).

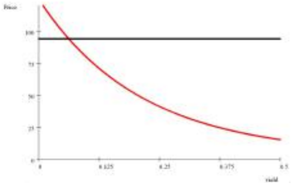


Figure 23: Inverse relationship between yield to maturity and bond price (red curve). The market bond price is the black line. The intersection point returns the bond yield to maturity on the horizontal axis

Example (YTM using annual compounding)

Let us consider the following cash flows on a coupon bond with semi-annual coupons and coupon rate of 4%.

Time (years)	0	0.5	1
CF	-98	2	102

The ytm solves the equation

$$98 = \frac{2}{(1+y)^{0.5}} + \frac{102}{(1+y)^{1}}$$

and setting $x = 1/(1+y)^{0.5}$, the equation becomes

$$98 = 2x + 102x^2,$$

equation having meaningful solution if $x = 0.9704412$, so that

$$y = \frac{1}{(0.9704412)^2} - 1 = 6.1846\%.$$

Example

Let us consider a bond that quotes at 99, expiring in 13 months and having semi-annual coupons and coupon rate of 6%. Coupon Dates will be in 1, 7 and 13 months. The Accrued Interest is $0.06/2 \times 5/6 \times 100 = 2.5$. The Gross Price is $99 + 2.5$. The bond cash flows are

Time	0	1m	7m	13m
CF	-101.5	3	3	103

The ytm solves the equation

$$101.5 = \frac{3}{(1+y)^{\frac{1}{12}}} + \frac{3}{(1+y)^{\frac{7}{12}}} + \frac{103}{(1+y)^{\frac{13}{12}}}.$$

This equation cannot be solved in closed form and needs some numerical procedure (such as bisection or Newton method, implemented in Matlab and Excel via fzero and the Solver). The ytm is 7.0887%.

Example (YTM using semi-annual compounding)

Let us consider the following cash flows on a coupon bond with semi-annual coupons and coupon rate of 4%. If we adopt the semi-annual compounding convention, y is the annual ytm and $y/2$ the semi-annual one and we measure time in semesters.

Time (semesters)	0	1	2
CF	-98	2	102

The ytm solves the equation

$$98 = \frac{2}{\left(1+\frac{y}{2}\right)^1} + \frac{102}{\left(1+\frac{y}{2}\right)^2}$$

and setting $x = 1/(1+\frac{y}{2})^1$, the equation becomes $98 = 2x + 102x^2$, It has as a meaningful solution $x = 0.9704412$, so that

$$y = 2\left(\frac{1}{x} - 1\right) = 2\left(\frac{1}{(0.9704412)} - 1\right) = 6.091827\%.$$

Meaning of the yield to maturity

- Yield is just a convenient way of expressing price and it is taken as a synthetic measure of bond's return.
- However, this interpretation rests on the assumption, absolutely flawed, that:

1. coupons will be reinvested at the initial YTM of the bond. Instead, the uncertainty of the future reinvestment rates makes uncertain the bond's return.
2. the bond is held to maturity.

- It is also a flawed concept because

a. it applies the same discount rate to different cash flows of the same bond;
b. it applies different discount rates to cash flows falling on the same date, but belonging to different bonds.

- Bonds with the same maturity will have different yields if their coupons differ and if the term structure is not flat. If one bond yields more than another it does not mean that it is of better value.
- It is actually inaccurate to compare bonds using the YTM.

Yield Curves

A yield curve is a graph of yield y against maturity (in years).

Figure 24: Yield Curves for US, ITA and GER. See Excel file FI_BasicYields.xlxm, Sheet: YTMUSDBond, YTMITABond, YTMGERBond.

Table 21: US Daily Treasury Yield Curve Rates. `https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield`

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
12/02/19	1.6	1.58	1.6	1.62	1.6	1.6	1.6	1.7	1.8	1.83	2.15	2.28
12/03/19	1.56	1.54	1.57	1.57	1.6	1.5	1.5	1.5	1.7	1.72	2.03	2.17
12/04/19	1.59	1.54	1.55	1.56	1.6	1.6	1.6	1.6	1.7	1.77	2.08	2.22
12/05/19	1.52	1.56	1.54	1.55	1.6	1.6	1.6	1.6	1.7	1.8	2.11	2.24
12/06/19	1.52	1.55	1.53	1.56	1.6	1.6	1.6	1.7	1.8	1.84	2.14	2.29
12/09/19	1.54	1.54	1.54	1.58	1.6	1.6	1.6	1.7	1.8	1.83	2.13	2.27
12/10/19	1.53	1.55	1.56	1.57	1.6	1.7	1.7	1.7	1.8	1.85	2.12	2.26
12/11/19	1.54	1.56	1.57	1.58	1.6	1.6	1.6	1.6	1.7	1.79	2.08	2.23
12/12/19	1.57	1.57	1.56	1.57	1.6	1.7	1.7	1.7	1.8	1.9	2.18	2.32
12/13/19	1.55	1.57	1.57	1.56	1.5	1.6	1.6	1.7	1.8	1.82	2.11	2.26
12/16/19	1.57	1.57	1.57	1.58	1.5	1.7	1.7	1.7	1.8	1.89	2.17	2.3
12/17/19	1.56	1.56	1.56	1.58	1.5	1.6	1.7	1.7	1.8	1.89	2.18	2.31
12/18/19	1.56	1.57	1.56	1.58	1.5	1.6	1.7	1.7	1.9	1.92	2.22	2.35

Conclusions

We have reviewed

- Main Compounding conventions
- Term Structure of discount factors
- Zero-coupon
- Coupon bond and pricing conventions

Appendix

US Treasury Bills

- TBills are US Treasury securities issued with maturities of 4, 13, 26 and 52 weeks.
- They do not pay coupons and are traded on the basis of **a discount to par**. The discount is sometimes referred to as a discount yield and should not be confused with a bond yield.
- Discount rates are quoted at an annual rate based on 360-day year for US and 365-day year for sterling instruments.

Fact (T-Bill quotations)

The percentage price paid for a money market instrument quoted at a discount rate d is:

$$P = 100 \times \left(1 - \frac{d}{100} \times \alpha_{t,T}\right)$$

where P is the percentage price, d the discount rate %, $\alpha_{t,T}$ the fraction of a year from settlement to redemption (ACT/360 or ACT/365).

Term	Issue Date	Maturity Date	Discount Rate %	Investment Rate %	Price Per $100	CUSIP
91-DAY	02-03-2005	05-05-2005	2.475	2.525	99.374375	912795SM4
182-DAY	02-03-2005	08-04-2005	2.710	2.786	98.629944	912795VK4
28-DAY	01-27-2005	02-24-2005	2.030	2.061	99.842111	912795SB8
91-DAY	01-27-2005	04-28-2005	2.320	2.366	99.413556	912795SL6
182-DAY	01-27-2005	07-28-2005	2.610	2.682	98.680500	912795VJ7
28-DAY	01-20-2005	02-17-2005	1.910	1.939	99.851444	912795SA0
91-DAY	01-20-2005	04-21-2005	2.360	2.407	99.403444	912795SK8
182-DAY	01-20-2005	07-21-2005	2.635	2.708	98.667861	912795VH1
5-DAY	01-13-2005	01-18-2005	2.040	2.069	99.971667	912795TG6
28-DAY	01-13-2005	02-10-2005	1.980	2.011	99.846000	912795RZ6

Figure 25: UST-Bills Auction Results. Source: `https://www.treasurydirect.gov/instit/annceresult/annceresult.htm`

Example (US T-Bill 91279SM4)

Issue Date		Feb. 2nd, 2005
Value Date	t	Feb 3rd, 2005
Maturity Date	T	May 5th, 2005
Time to maturity (ACT/360)	$days\,(t, T)$	91/360
Discount rate	d	$\frac{2.475}{100}$
Traded Price	$P\,(t, T)$	$100 \times \left(1 - \frac{2.475}{100} \times \frac{91}{360}\right) = 99.374$
Investment Rate		$\frac{100-99.374375}{99.374375} \times \frac{365}{91} = 2.5252\%$
Equiv. Yield		$\frac{100-99.374375}{99.374375} \times \frac{360}{91} = 2.49058\%$

The **Investment Rate** is computed according to the convention ACT/365 to compare the return on TB to the one in the bond market

$$IR = \frac{d}{1 - d \times \frac{ACT}{360}} \times \frac{365}{360}.$$

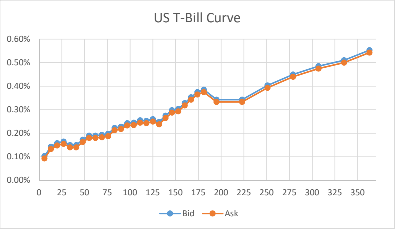
Similarly, to make it comparable to money-market rates, we can also compute the so called **Money Market Equivalent Yield**

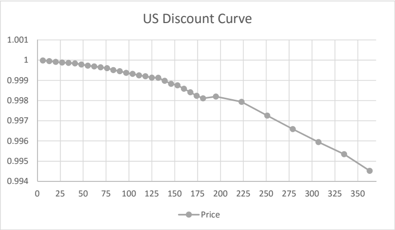
$$EY = \frac{d}{1 - d \times \frac{ACT}{360}}.$$

Trade	April 29th, 2016								
Maturity	Days	Bid	Ask	Price	Maturity	Days	Bid	Ask	Price
05/05/2016	6	0.103%	0.093%	0.999985					
12/05/2016	13	0.143%	0.133%	0.999952	01/09/2016	125	0.260%	0.250%	0.999132
19/05/2016	20	0.158%	0.148%	0.999918	08/09/2016	132	0.248%	0.238%	0.999127
26/05/2016	27	0.165%	0.155%	0.999884	15/09/2016	139	0.275%	0.265%	0.998977
02/06/2016	34	0.150%	0.140%	0.999868	22/09/2016	146	0.298%	0.288%	0.998832
09/06/2016	41	0.150%	0.140%	0.999841	29/09/2016	153	0.303%	0.293%	0.998755
16/06/2016	48	0.173%	0.163%	0.999783	06/10/2016	160	0.328%	0.318%	0.998587
23/06/2016	55	0.190%	0.180%	0.999725	13/10/2016	167	0.353%	0.343%	0.998409
30/06/2016	62	0.190%	0.180%	0.99969	20/10/2016	174	0.375%	0.365%	0.998236
07/07/2016	69	0.193%	0.183%	0.999649	27/10/2016	181	0.385%	0.375%	0.998115
14/07/2016	76	0.198%	0.188%	0.999603	10/11/2016	195	0.343%	0.333%	0.998196
21/07/2016	83	0.223%	0.213%	0.999509	08/12/2016	223	0.343%	0.333%	0.997937
28/07/2016	90	0.228%	0.218%	0.999455	05/01/2017	251	0.403%	0.393%	0.99726
04/08/2016	97	0.243%	0.233%	0.999372	02/02/2017	279	0.450%	0.440%	0.99659
11/08/2016	104	0.245%	0.235%	0.999321	02/03/2017	307	0.485%	0.475%	0.995949
18/08/2016	111	0.255%	0.245%	0.999245	30/03/2017	335	0.510%	0.500%	0.995347
25/08/2016	118	0.253%	0.243%	0.999204	27/04/2017	363	0.553%	0.543%	0.994525

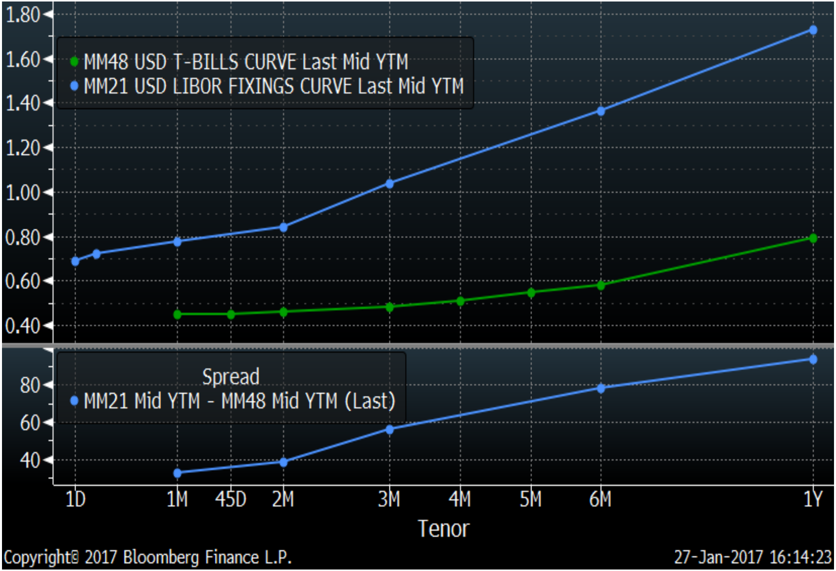
Table 22: US short term government curve (Secondary Market). Quotes available on the Wall Street Journal webpage at `http://online.wsj.com/mdc/public/page/2_3020-treasury.html#treasuryB` and `https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=billrates`

For greater details on the pricing of T-Bills: `https://developers.opengamma.com/quantitative-research/Bill-Pricing-OpenGamma.pdf`


US T-Bill Curve
0.60%
0.50%
0.40%
0.30%
0.20%
0.10%
0.00%
0
25
50
75
100
125
150
175
200
225
250
275
300
325
350
Bid
Ask


US Discount Curve
1.001
1
0.999
0.998
0.997
0.996
0.995
0.994
0
25
50
75
100
125
150
175
200
225
250
275
300
325
350
Price

Treasury Bill Rate and LIBOR rate



Historical Spread LIBOR Rate - TB Rate

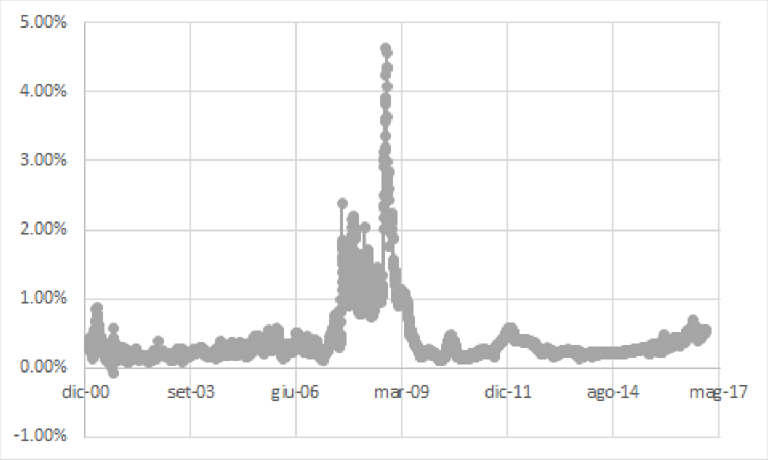


Figure 26: Spread 3m US LIBOR vs 3m T-Bill Equivalent Yield.

Rolls Dates for Odd Coupon Bonds I

- Sometimes the rolls dates are not straightforward.
- What about the roll dates for a bond that is issued on 1-Apr-2016 and matures on 1-June-2026?
- This case refers to the so called Odd Coupons
- At first, we need to determine when the stub or odd coupon occurs.
- Normally, the stub is at the front and then roll dates are determined from and including the maturity date;

Rolls Dates for Odd Coupon Bonds II

Example (Odd Coupon at the front)

Issue Date	01/04/2016
Maturity Date	01/06/2019
First Coupon Date	**01/06/2016**
Frequency	2
Coupon	5

Times	01/06/2016	01/12/2016	···	···	01/06/2019
Days	**61**	183	182	183	182
Cash Flow	2.5	2.5	···	···	102.5

Rolls Dates for Odd Coupon Bonds III

- Alternatively, a bond has an odd last coupon, if the final payment for the bond occurs on a date that is out of synch with the rest of the coupon dates.

Example (Odd Coupon at the end)

Issue Date	01/04/2016
Maturity Date	01/06/2019
Last Coupon Date	**01/10/2018**
Frequency	2
Coupon	5

Times	01/10/2016	$\cdots$	01/10/2018	01/04/2019	01/06/2019
Days	183	$\cdots$	183	182	61
Coupons	2.5	$\cdots$	2.5	**0**	102.5

Yield to Maturity

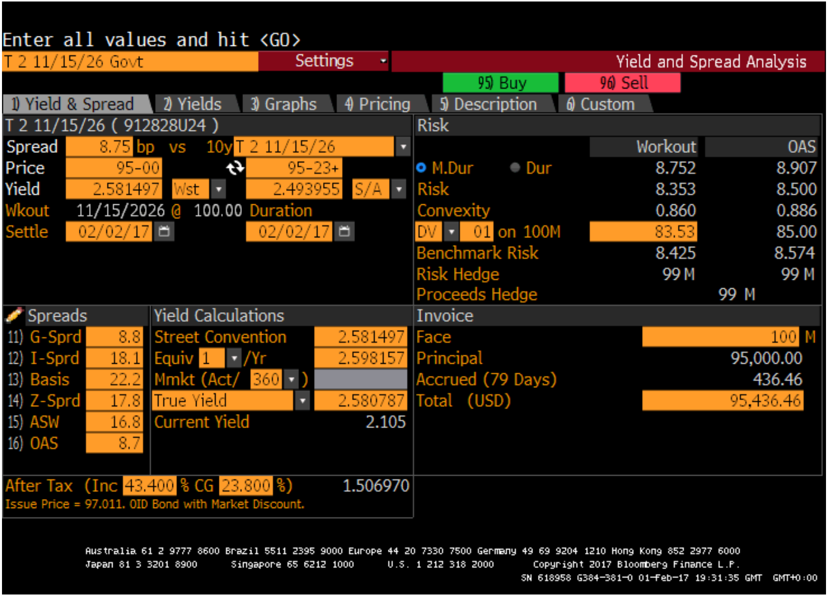


Figure 27: Bllomberg page with ytm computation

Computing the Yield to maturity: True Yield I

True Yield

- True yields are computed based on a cash flow stream that has been adjusted for the actual payment dates.
- That is, each scheduled coupon payment is discounted back from the actual payment date based on the selected business day convention and the relevant holiday schedule provided by the user.

$$B\left(c, t; t_1, ..., t_n\right) = \sum_{i=1}^{n} \frac{c/m}{\left(1+y\right)^{t_i - t}} + \frac{1}{\left(1+y\right)^{t_n - t}},$$

- The year fractions $t_i - t$ are measured in years and adjustments for weekend and holidays are taken in account.
- It is applicable to bonds with no call or put provisions.

Computing the Yield to maturity: True Yield II

	A	B	C	D	E	F	G
1	**Yield to maturity for a BTP**						
2	**BTP 01.02.2006**						
3	ISIN	IT0003424485					
4	**BOND INFORMATION**						
5	Face Value	100					
6	Annual coupon	2.75					
7	Maturity	01/02/2006					
8	Basis	1					
9	Number of coupons in the year	2					
10	**MARKET INFORMATION**						
11	Trade Date	28/10/2003					
12	Clean Price	99.76000					
13	**BOND CALCULATOR**						
14	Value Date	31/10/2003	=WORKDAY(B11;3;$J$2:$J$363)		Cell B14 has been named ValueDate		
15	Payment of last coupon	01/08/2003	=COUPPCD(ValueDate;B7;B9;B8)				
16	Date of next coupon	01/02/2004	=COUPNCD(ValueDate;B7;B9;B8)				
17	Number of remaining coupons	5	=COUPNUM(ValueDate;B7;B9;B8)				
18	Days since last coupon	91	=COUPDAYBS(ValueDate;B7;B9;B8)				
19	Days to next coupon	184	=COUPDAYS(ValueDate;B7;B9;B8)				
20	Accrued Interest	0.68003	=ROUND((B6/2)*B18/(B19);5)				
21	Market Gross Price	100.44003	=B12+B20				
22	**YIELD CALCULATOR**						
23	Yield to Maturity	**2.8738%**					
24	Theoretical Price	100.440022	=SUM(G29:OFFSET($G$29;B17;0))				
25	Pricing Error	8.09406E-06	=ABS(B24-B21)				

Computing the Yield to maturity: True Yield III

	B	C	D	E	F	G
21	Cash Flows Details					
22	Coupon Dates	Adjusted	Days	Semiannual	Face Value	Present Value
23		Coupon Dates		Coupon		
24	Sunday, 1 February 2004	02/02/2004	94	1.375	0	1.36500
25	Sunday, 1 August 2004	02/08/2004	276	1.375	0	1.34586
26	Tuesday, 1 February 2005	01/02/2005	459	1.375	0	1.32687
27	Monday, 1 August 2005	01/08/2005	640	1.375	0	1.30836
28	Wednesday, 1 February 2006	01/02/2006	824	1.375	100	95.09393

Figure 28: Computing the true yield to maturity for an Italian BTP.

Computing the Yield to maturity: US Street Yield I

US Street Yield

- The convention used by market participants in the U.S. to value treasuries.
- Based on an accrual basis of Actual/Actual and assumes that yields are compounded semi-annually, even in fractional first periods (compare with the U.S. Treasury method).
- If the bond is in its final coupon period, then the US Street yield is computed using the US market final period pricing convention (see money market yield).

Computing the Yield to maturity: US Street Yield II

It is applicable to bonds with:

- Fixed coupon payments that do not vary as a function of the actual dates between payments (with the possible exception of the first and last payments).
- No allowance for the precise timing of the cash flows. That is, bonds are conventionally priced by assuming that each coupon payment falls on the nominal payment date with no adjustment for holidays or weekends.
- A single redemption date and a fixed redemption value (bullet bonds).
- No call or put provisions.

Computing the Yield to maturity: US Street Yield III

US Treasury Yield (ISMA formula)

- The convention used by the U.S. Treasury to value bonds.
- Based on an accrual basis of Act/Act and assumes that yields are compounded semi-annually in all but the fractional first period.
- If the bond is in its final coupon period, then the US Street yield is computed using the US market final period pricing convention (see money market yield).

Same caveats as in the previous case.

ISMA formula for the YTM (US Street yield) I

The following bond pricing method is based on the ISMA redemption yield formula:

$$B\left(c, t; t_1, ..., t_n\right) = \sum_{i=1}^{2} \frac{c/m}{\left(1 + \frac{y_2}{2}\right)^{t_i - t}} + \frac{1}{\left(1 + \frac{y_2}{2}\right)^{t_{2n} - t}},$$

where n is the number of remaining semi-annual coupons and the times $t_i - t$ are measured in semesters.

ISMA formula for the YTM (US Street yield) II

The above formula can be simplified in the following way:

- Define

$$
\begin{aligned}
n &= \text{Numbers of remaining coupons} \\
u &= \text{Days Since Last Coupon} \\
v &= \text{Days Until Next Coupon} \\
w &= \frac{v}{u+v} \\
\phi &= \frac{1}{1+\frac{y_m}{m}}
\end{aligned}
$$

then y solves

$$B\left(c,t;t_1,...,t_n\right) = \phi^w \left(\frac{c}{m}\phi\frac{1-\phi^{n-1}}{1-\phi} + \frac{c}{m} + \phi^{n-1}\right)$$

ISMA formula for the YTM (US Street yield) III

- y_2 is called the **bond-equivalent yield**.
- The effective rate (comparable to the true yield) is

$$\left(1+\frac{y_2}{2}\right)^2-1.$$

- This formula is also known as **Brass-Fangmeyer method** (German Bunds).

ICMA formula for the YTM (US Street yield)

The following bond pricing method is based on the ISMA redemption yield formula:

$$TB\left(y; t, t_1, ..., t_n\right) = \frac{1}{1 + w \times \frac{y}{2}} \sum_{i=1}^{2n} \frac{c/m}{\left(1 + \frac{y_2}{2}\right)^{t_i - t}} + \frac{1}{\left(1 + \frac{y_2}{2}\right)^{t_{2n} - t}},$$

and where the time $t_i - t$ are measured in semesters. Using the previous notation, then y solves

$$B\left(c, t; t_1, ..., t_n\right) = \frac{1}{1 + \frac{y_m}{m} \times w} \left(\frac{c}{m}\phi \frac{1 - \phi^{n-1}}{1 - \phi} + \frac{c}{m} + \phi^{n-1}\right)$$

The effective rate is

$$\left(1 + \frac{y_2}{2}\right)^2 - 1.$$

This formula is also known as **Moosmuller method** (German Bunds).

US BOND			
ISIN	912828U24		
BOND INFORMATION			
Face Value	100		
Annual coupon	2.875		
Maturity	15/11/2046		
Basis	1		
Number of coupons in the year	2		
MARKET INFORMATION			
Trade Date	01/02/2017		
Clean Price	96.00780		
BOND CALCULATOR			
Value Date	02/02/2017		=WORKDAY(B11;1;)
Payment of last coupon	15/11/2016		=COUPPCD(ValueDate;B7;B9;B8)
Date of next coupon	15/05/2017		=COUPNCD(ValueDate;B7;B9;B8)
Number of remaining coupons	60	n	=COUPNUM(ValueDate;B7;B9;B8)
Days since last coupon	79	u	=COUPDAYBS(ValueDate;B7;B9;B8)
Days to next coupon	181	v	=COUPDAYS(ValueDate;B7;B9;B8)
Accrued Interest	0.62742		=ROUND((B6/2)*B18/(B19);5)
Market Gross Price	96.63522		=B12+B20

YIELD CALCULATOR			
	US Street	ISMA Method	
Bond Equivalent Yield	**3.0703%**	**3.0702%**	
ϕ	0.98488	0.98488	=1/(1+B24/$B$9)
w	0.69615	0.69615	=$B$19/($B$19+$B$18)
Annuity	38.62585	38.62646	=B25*(1-B25^(B17-1))/(1-B25)
Theoretical Pricev in t1	97.66557	97.66792	=(B6/B9)*(1+B27)+B5*B25^(B17-1)
Theoretical Pricev in t	96.63522	96.63522	=B28*B25^B26
Pricing Error	0.0000	0.0000	=ABS(B29-B21)
Effective Yield	**3.0939%**	**3.0937%**	=(1+B24/$B$9)^$B$9-1

Figure 29: Computing yield to maturity according to different conventions.

Example (1. YTM for a Treasury Bond)

- Trade Date: 28-Apr-2016.
- Settlement Date: 29-Apr-2016.
- 30-Nov-2017 Treasury: Ask Price 99.8438, Coupon 0.625, Quoted yield: 0.724%.
- Coupon Dates: 30-Nov and 30-May.
- Days in the coupon Period: 183
- Days since last coupon (u): 151
- Days to next coupon (v): 32
- Number of remaining coupons (n): 4
- Accrued Interest

$$\frac{0.625}{2}\frac{151}{183} = 0.2579$$

- Invoice Price

$$99.8438 + 0.2579 = \mathbf{100.1017}.$$

Example (2. YTM for a Treasury Bond (ISMA Method))

- We have

$$w = \frac{v}{u+v} = \frac{32}{151+32} = 0.17486.$$

- Discount factor (semi-annual)

$$\phi = \frac{1}{1+\frac{0.00724}{2}} = 0.99639.$$

- Annuity

$$\phi\frac{1-\phi^{n-1}}{1-\phi} = 2.97841.$$

- Discount factor for short period

$$\phi^w = 0.99639^{0.17486} = 0.99937.$$

- Present value of coupon payments

$$\frac{0.625}{2} \times (2.97841+1) \times 0.99937 = 1.24247.$$

- Present value of principal

$$100 \times \phi^{n-1+w} = 98.85933$$

- Theoretical Price

$$1.24247 + 98.85933 = \mathbf{100.1018}.$$

Nominal Yield Spread I

- The `yield spread` displays the yields of coupon-bearing bonds as function of time to maturity. The `absolute yield spread` between any two bond issues, bond X and bond Y, is computed as follows:

$$\text{yield spread} = \text{yield on bond}_X - \text{yield on bond}_Y.$$

- This traditional yield spread is also known as the **nominal spread**.

Nominal Yield Spread II

Table 23: Nominal Spreads Germany-Italy on 30-Apr-2016

Years	Days	ITA	GER	Nominal Spread
1	365	-0.21%	-0.65%	0.44%
2	730	-0.05%	-0.54%	0.49%
3	1095	0.07%	-0.55%	0.62%
4	1460	0.22%	-0.51%	0.72%
5	1825	0.47%	-0.40%	0.87%
6	2190	0.69%	-0.34%	1.03%
7	2555	0.88%	-0.20%	1.08%
8	2920	1.16%	-0.18%	1.33%
9	3285	1.31%	-0.07%	1.38%
10	3650	1.44%	0.04%	1.40%
15	5475	1.94%	0.82%	1.12%
20	7300	2.23%	0.92%	1.32%

Nominal Yield Spread III

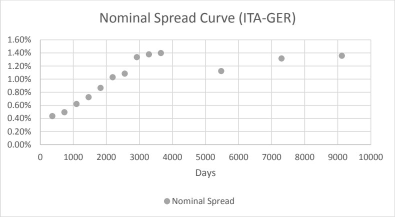


Figure 30: Nominal Spread Italy-Germany

- In practice, this plot is informative but of very limited practical use.
- There is no reason to expect the credit spread to be the same regardless of when the cash flow is received.

Bond Portfolio Yield

- The yield for a portfolio of bonds is found by solving the rate that will make the present value of the portfolio's cash flows equal to the market value of the portfolio.
- For example, a portfolio consisting of a two-year, 5% annual coupon bond priced at par (100) and a three-year, 10% annual coupon bond priced at 107.87 to yield 7% (YTM) would generate a three-year cash flow of \$15, \$115, and \$110 and would have a portfolio market value of \$207.87.
- The rate that equates this portfolio's cash flow to its portfolio value is 6.2%. Indeed it solves the equation

$$207.87 = \frac{15}{(1+y)^1} + \frac{115}{(1+y)^2} + \frac{110}{(1+y)^3}.$$

- The bond portfolio yield is not the weighted average of the YTM of the bonds comprising the portfolio. In this example, the weighted average (Rp) is

$$6.04\% = (0.05 \times 100/207.87 + 0.07 \times 107.87/207.87).$$

Simple compounded spot rates: $L(t, T)$

- $L(t, T)$ is the constant rate at which an investment of $P(t, T)$ units at time t accrues to yield a unit amount at maturity T, given that the amount accrued is proportional to the investment length.
- $L(t, T)$ is then the solution of:

$$P(t, T)\left(1 + L(t, T)\,\alpha_{t,T}\right) = 1,$$

We have:

Simple Compounding/Discounting

- Given the discount factor, we can compute the **simple** compounded interest rate

$$L(t, T) = \frac{1}{\alpha_{t,T}}\left(\frac{1 - P(t, T)}{P(t, T)}\right).$$

- Viceversa, given $L(t, T)$, the zcb price is

$$P(t, T) = \frac{1}{1 + L(t, T)\,\alpha_{t,T}}.$$

Annualy compounded spot rates: $Y(t, T)$

- $Y(t, T)$ is the constant rate at which an investment of $P(t, T)$ units at time t accrues to yield a unit amount at maturity T, given that the interest obtained is reinvested once a year.
- $Y(t, T)$ is then the solution of:

$$P(t, T)\left(1 + Y(t, T)\right)^{\alpha_{t,T}} = 1,$$

We have:

Annualy Compounding/Discounting

- Given the discount factor, we can compute the **annually** compounded interest rate

$$Y(t, T) = \left(\frac{1}{P(t, T)}\right)^{\frac{1}{\alpha_{t,T}}} - 1.$$

- Viceversa, given $Y(t, T)$, we can compute the zcb price:

$$P(t, T) = \frac{1}{\left(1 + Y(t, T)\right)^{\alpha_{t,T}}}.$$

Continuously compounded spot rate: $R(t, T)$

- $R(t, T)$ is the constant rate at which an investment of $P(t, T)$ units at time t `accrues continuously` to yield a unit amount at maturity T.
- $R(t, T)$ is then the solution of:

$$P(t, T)\, e^{R(t,T)\alpha_{t,T}} = 1.$$

We have:

Continuous Compounding/Discounting

- Given the discount factor, we can compute the **continuously** compounded interest rate

$$R(t, T) = -\frac{\ln P(t, T)}{\alpha_{t,T}}.$$

- Viceversa, given the continously interest rate, we can compute the zcb price in terms of $R(t, T)$:

$$P(t, T) = e^{-R(t,T)\alpha_{t,T}}.$$

A strange animal: the instantaneous interest rate $r(t)$

- The instantaneous interest rate, or `short rate`, is the return on a spot deposit of infinitesimal lenght.
- $r(t)$ is defined as the limit of $R(t, T)$ (or of $L(t, T)$) as $T \to t$:

$$r(t) = \lim_{T \to t} R(t, T) = -\left.\frac{\partial \ln P(t, T)}{\partial T}\right|_{T=t}.$$

- Remarks:
  1. It is a convenient quantity to use for modelling purposes: the math is simpler. For this reason, the first term structure models (Vasicek and CIR) were short rate models, i.e. models assigning the dynamics of $r(t)$.
  2. Note that $r(t)$ does not depend on the maturity T any longer.
  3. $r(t)$ represents just a point on the term structure of spot rates: it is the intercept on the vertical axis.
  4. $r(t)$ does not exist as traded quantity in the market.

$r(t)$ and $P(t,T)$

- In general, we cannot recover the whole zero-rate curve by knowing the short rate at time t only.
- Indeed, given the discount curve we can obtain $r(t)$ according to the previous formula

$$P(t,T), \forall T \quad \Rightarrow \quad r(t).$$

- Viceversa, given the short rate we cannot recover the discount curve

$$r(t) \quad \nRightarrow \quad P(t,T).$$

- However, assuming a future deterministic path for $r(.)$, i.e. if we know the entire future path $r(s)$, $t \leq s \leq T$, then

$$P(t,T) = e^{-\int_t^T r(s)ds}.$$

- In general, outside the deterministic world, this relationship is not true.
- In order to understand the relatioship between P and r, we need to remember some basic fact on no-arbitrage pricing. In particular, we need to remember the concept of `money market account`, `risk-neutral expectation` and `martingale`.

$f(t,T)$ and $r(t)$

- Let us consider $r(t)$. By definition:

$$r(t) = \lim_{T \to t} R(t,T) = \lim_{T \to t} \frac{\int_t^T f(t,s)ds}{T-t} = f(t,t)$$

  i.e. the instantaneous interest rate is a particular forward rate.
- The knowledge of $r(t)$ does not say anything about $f(t,T)$, $T > t$.
- Only if we known the future path of $r(.)$, we can recover $f(t,T)$:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = \frac{\partial \int_t^T r(s)\,ds}{\partial T} = r(T)$$

- Then, `under a deterministic evolution`, the instantaneous interest rate at time T is equal to the current instantaneous forward rate $f(t,T)$.

Continuously compounded forward rates

Continuously Compounded Forward Rate

- With continuously compounded rates, we define $F(t, T_1, T_2)$ such as:

$$\frac{P(t, T_2)}{P(t, T_1)} e^{F(t, T_1, T_2)(T_2 - T_1)} = 1,$$

- Therefore the Continuously Compounded Forward Rate is

$$F(t, T_1, T_2) = -\frac{\ln P(t, T_2) - \ln P(t, T_1)}{T_2 - T_1}.$$

- Assuming c.c., express $F(t, T_1, T_2)$ in terms of spot rates and show that:

$$F(t, T_1, T_2) = \frac{(T_2 - t) R(t, T_2) - (T_1 - t) R(t, T_1)}{T_2 - T_1}.$$

- Repeat using simple convention.

- Discount rates are used in the T-bill spot and futures markets.
- The discount rate is not a rate of return, because it is calculated as a percentage of the final value of the bill, not the initial investment.

Discount rate

The discount rate $d(t, T)$ is defined by:

$$d(t, T) = \frac{1}{\alpha_{t,T}} (1 - P(t, T)),$$

where $\alpha_{t,T}$ follows the day-count convention ACT/360 (US TBills) or ACT/365 (UK sterling).
We can express also the zcb price in terms of $d(t, T)$:

$$P(t, T) = 1 - d(t, T)\, \alpha_{t,T}$$

Bond Equivalent Yield or Investment Rate

- The measure that seeks to make the Treasury bill quote comparable to coupon Treasuries is called the `bond-equivalent yield`.
- It is computed according to the convention ACT/365:

$$BEY = \frac{d}{1 - d \times \frac{ACT}{360}} \times \frac{365}{360}.$$

Example (Computing the Investment Rate)

$$\frac{\frac{2.475}{100}}{1 - \frac{2.475}{100} \times \frac{91}{360}} \times \frac{365}{360} = 2.525\,2\%.$$

CD equivalent yield

- The `CD equivalent yield` (also called the money market equivalent yield) makes the quoted yield on a Treasury bill more comparable to yield quotations on other money market instruments that pay interest on a 360-day basis:

$$CDEY = \frac{d}{1 - d \times \frac{ACT}{360}}.$$

Example

Given the 2.475% T-Bill yield, the CD equivalent yield is

$$\frac{\frac{2.475}{100}}{1 - \frac{2.475}{100} \times \frac{91}{360}} = 2.4906\%.$$

Rate of Return: Common Measures

- **Current Yield**: is the ratio of its annual coupon to its closing price.

$$C^Y = \frac{\text{Annual Coupon}}{B(c,t)}.$$

- **Coupon Rate**: is the contractual rate the issuer agrees to pay each period. It is expressed as a proportion of the annual coupon payment to the bond's face value:

$$C^R = \frac{\text{Annual Coupon}}{F}.$$

- **Yield to Maturity**: is the rate that equates the price of the bond, B, to the PV of the bond's cash flow (CF); it is the internal rate of return, IRR, of a bond. It can be computed according to different conventions.

JGB Yield

- The yield convention used by Japanese Government Bonds.
- Yield is based on the following simple interest formula:

$$ytm_{simple} = \frac{c + (100 - (B - AI))/T}{(B - AI)/100}$$

where T is the number of days from settle to maturity divided by 365.

- If there is more than one year to maturity, the numerator must be reduced for every leap day that falls within the period. Notice that in the formula the clean price is used.
- With reference to the bond under examination, we have

$$ytm_{simple} = \frac{2.75 + (100 - 99.76)/2.2575}{99.76/100}$$

Main relationships: Simple Compounding

Simple compounding			
	$P(t,T)$	$L(t,T)$	$F(t,T_1,T_2)$
$P(t,T)$	-	$\frac{1}{1+L(t,T)(T-t)}$	$\frac{1}{1+F(t,T,T)(T-t)}$
$L(t,T)$	$\frac{1}{T-t}\left(\frac{1}{P(t,T)}-1\right)$	-	$F(t,T,T)$
$F(t,T_1,T_2)$	$\frac{1}{\tau}\left(\frac{P(t,T_1)}{P(t,T_2)}-1\right)$	....	-
$r(t)$	$-\left.\frac{\partial \ln P(t,T)}{\partial T}\right\vert_{T=t}$	$L(t,T)\vert_{T=t}$	$F(t,T,T)\vert_{T=t}$

Day Count Conventions

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Outline

References

Useful Readings

- OpenGamma Interest Rate Instruments and Market Conventions Guide. 16th December 2013. Available at: https://developers.opengamma.com/quantitative-research/Interest-Rate-Instruments-and-Market-Conventions.pdf
- 30/360 Day Count Conventions. Excel spreadsheet with worked 30/360 examples available at http://www.isda.org/c_and_a/trading_practice.html.
- Euro market ACT/ACT Day Count Conventions: available at http://www.isda.org/c_and_a/trading_practice.html.
- Wikipedia at https://en.wikipedia.org/wiki/Day_count_convention

Excel Files

- 30-360-2006ISDADefs.xls

Day-Count Convention

Day-Count convention

- The particular choice that is made to measure the time between two dates is known as DAY-COUNT CONVENTION.
- The bond markets in Europe and elsewhere have developed independently with different conventions for calculating prices, yields and interest rates and settling the various instruments. Recent changes are making the markets more homogeneous, due also to the advent of the Euro and the wide acceptance of the ISMA yield methodology.
- The most frequently used day-count conventions are (see James and Webber, pagg. 52-53):
    1. 30/360
    2. Actual/365
    3. Actual/360
- Different countries and markets handle in different way exceptional situations.

Day-Count convention: 30/360

30/360

- This accrual method assumes 30 days per month and 360 days per year. Hence, the accrual factor is simply the number of accrued days divided by 360.
- The 30/360 method groups a certain number of methods that have in common the accrual factor as

$$\frac{360 \times (Y_2 - Y_1) + 30 \times (M_2 - M_1) + (D_2 - D_1)}{360}$$

  but differs on how the Y_i, M_i and D_i are computed.
- In practice, we need to count the whole number of calendar months in between the two dates and then add on the fractions of each month at the start and end of the period.

30/360 ISDA

3. 30/360

This is definition 4.16(f) in 2006 ISDA Definitions. The date adjustment rules are the following:

- If D1 is 31, then change D1 to 30.
- If D2 is 31 and D1 is 30 or 31, then change D2 to 30.

This day count convention is also called *30/360 US*, *30U/360*, *Bond basis*, *30/360* or *360/360*. The last three terms are the ones used in the 2006 ISDA Definitions.

There exists also a version of the day count which depends on an EOM convention. In that case an extra rule is added:

- If EOM and D1 is last day of February and D2 is last day of February, then change D2 to 30 and D1 to 30.

The ISDA definitions do not refer to the EOM convention.

Other names: Bond Basis, 30-360 U.S. Municipal

Example
The year fraction between January 31st, 2007 and February 28th, 2007 is:

$$D_1 = 30, D_2 = 28; M_1 = 1, M_2 = 2; Y_1 = 2007, Y_2 = 2007,$$

therefore

$$\frac{360 \times 0 + 30 \times 1 + (28 - 30)}{360} = \frac{28}{360}.$$

30E/360 ISDA and 30E/360 ISDA

4. 30E/360

This is definition 4.16(g) in 2006 ISDA Definitions. The date adjustment rules are the following:

- If D1 is 31, then change D1 to 30.
- If D2 is 31, then change D2 to 30.

This day count convention is also called *Eurobond basis*.

5. 30E/360 (ISDA)

This is definition 4.16(h) in 2006 ISDA Definitions. The date adjustment rules are the following:

- If D1 is the last day of the month, then change D1 to 30.
- If D2 is the last day of February but not the termination date or D2 is 31, then change D2 to 30.

Example 30E/360
The year fraction between Sept 30th, 2007 and October 31st, 2008 is:

$$D_1 = 30, D_2 = 30; M_1 = 9, M_2 = 10; Y_1 = 2007, Y_2 = 2007,$$

therefore

$$\frac{360 \times 0 + 30 \times 1 + (30 - 30)}{360} = \frac{30}{360}.$$

Example 30E/360 (ISDA)
The year fraction between January 31st, 2007 and February 28th, 2007 is:

$$D_1 = 30, D_2 = 30; M_1 = 1, M_2 = 2; Y_1 = 2007, Y_2 = 2007,$$

therefore

$$\frac{360 \times 0 + 30 \times 1 + (30 - 30)}{360} = \frac{28}{360}.$$

Table 1: Sample calculations under alternative versions of 30/360 and 30E/360 found in 2006 ISDA Definitions. Source: David Mengle, ISDA Head of Research.

Calculation Period		**30/360 (Bond Basis)**			**30E/360 (Eurobond)**			**30E/360 (ISDA)**		
Start Date	End Date	D1	D2	Days	D1	D2	Days	D1	D2	Days
01/15/07	01/30/07	15	30	15	15	30	15	15	30	15
01/15/07	02/15/07	15	15	30	15	15	30	15	15	30
01/15/07	07/15/07	15	15	180	15	15	180	15	15	180
09/30/07	03/31/08	30	30	180	30	30	180	30	30	180
09/30/07	10/31/07	30	30	30	30	30	30	30	30	30
09/30/07	09/30/08	30	30	360	30	30	360	30	30	360
01/15/07	01/31/07	15	31	16	15	30	15	15	30	15
01/31/07	02/28/07	30	28	28	30	28	28	30	30	30
02/28/07	03/31/07	28	31	33	28	30	32	30	30	30
08/31/06	02/28/07	30	28	178	30	28	178	30	30	180
02/28/07	08/31/07	28	31	183	28	30	182	30	30	180
02/14/07	02/28/07	14	28	14	14	28	14	14	30	16
02/26/07	02/29/08	26	29	363	26	29	363	26	30	364

Day-Count conventions: Act/365

Actual/365 Fixed

- A year is 365 days long and the year fraction between two dates is the actual number of days between them divided by 365.
- The accrual factor is the actual number of accrued days divided by 365:

$$\frac{D_2 - D_1}{365}$$

  where $D_2 - D_1$ is the number of days between the two dates.
- The number 365 is used even in a leap year.
- This convention is called English Money Market basis.
- This accrual is sometimes used in the money market and in calculating accrued interest of bonds or swaps.

Example
The year fraction between February 5, 2002 and March 24, 2002 is:

$$\frac{47}{365} = 0.12877 \text{ years}$$

Actual/365 A

- A year is 365 days long and the year fraction between two dates is the actual number of days between them divided by 365.
- The accrual factor is the actual number of accrued days divided by 365:
$$\frac{D_2 - D_1}{Denominator}$$
where $D_2 - D_1$ is the number of days between the two dates.
- Denominator is 366 if 29 February is between D_1 (exclusive) to D_2 (inclusive) and 365 otherwise.
- The convention is also called ACT/365 Actual.

Day-Count convention: Act/360

Actual/360

- This accrual method calculates the actual number of days between two dates and assumes a year basis of 360 days.
- The accrual factor is the actual number of accrued days divided by 360

$$\frac{D_2 - D_1}{360}$$

where $D_2 - D_1$ is the number of days between the two dates.
- This is the most used day count convention for money market instruments (maturity below one year).
- This day count is also called Money Market basis, Actual 360, or French.

Example
The year fraction between February 5, 2002 and March 24, 2002 is:

$$\frac{47}{360} = 0.13056 \text{ years}$$

Day Count Convention: Act/Act

Act/Act ISDA

- The accrual factor is

$$\frac{\text{Days in a non-leap year}}{365} + \frac{\text{Days in a leap year}}{366}$$

- To compute the number of days, the period first day is included and the last day is excluded (1991 ISDA definitions).
- Examples:
  - Start date: 30-Dec-2010; End date: 2-Jan-2011:

$$\frac{3}{365} = 0.008219.$$

  - Start date: 30-Dec-2011; End date: 2-Jan-2012:

$$\frac{2}{365} + \frac{1}{366} = 0.8211.$$

  - Start date: 30-Dec-2010; End date: 2-Jan-2013:

$$\frac{367}{365} + \frac{366}{366} + \frac{1}{365} = 2.008219.$$

Act/Act ISMA and Act/Act AFB i

- The accrual factor according is

$$\frac{\text{Days in the period}}{\textit{Denominator}}$$

- Days in the period: actual number of days from and including the last coupon date to, but excluding, the current value date.
- ISMA: Denominator is the actual number of days in the coupon period multiplied by the number of coupon periods in the year.
- AFB: Denominator is either 365 (if the calculation period does not contain 29th February), or 366 (if the calculation period included 29th February).

Act/Act ISMA and Act/Act AFB ii

Examples

- Start date: 1st-Nov-2003; End date: 1st-May-2004:
- Days from Nov to 31 Dec: 61; Days from 31 Dec to May: 121.
- ISDA Method:

$$\frac{61}{365} + \frac{121}{366} = 0.497724.$$

- ISMA Method:

$$\frac{182}{182 \times 2} = 0.5.$$

- AFB Method:

$$\frac{182}{366} = 0.49727.$$

Other conventions

- 30E/360 ISDA
- NL/365
- Business/252
- For additional details, consult

a. https://en.wikipedia.org/wiki/Day_count_convention

b. https://wiki.treasurers.org/wiki/Day_count_conventions

Day Count Conventions across the World

	Money Market Basis	Government Bond Basis
Australia	Act/366	Act/Act
Canada	Act/365	Act/365
Euro	Act/360	Act/Act
Japan	Act/360	Act/365
UK	Act/365	Act/Act
USA	Act/360	Act/Act

Table 2: Daycount Basis for the major currencies

Excel function: YearFrac

- This function determines the fraction of a year occurring between two dates based on the number days between those dates using a specified day count basis.
- It is not very reliable.

	A	B	C	D	E	F	G	H
1	YearFrac(start_data;end_data;basis)							
2								
3								
4				d_e	10/25/1996	1/27/1998		
5				d_t	12/31/1996	2/1/1999		
6		Day Count	Basis					
7		30/360	0		0.183333333	1.011111111	=yearfrac($E$4,$E$5,C7)	
8		ACT/ACT	1		0.183060109	1.013698630	=yearfrac($E$4,$E$5,C8)	
9		ACT/360	2		0.186111111	1.027777778	=yearfrac($E$4,$E$5,C9)	
10		ACT/365	3		0.183561644	1.013698630	=yearfrac($E$4,$E$5,C10)	
11		30/360	4		0.180555556	1.011111111	=yearfrac($E$4,$E$5,C11)	
12								
13								

Matlab function: YearFrac

[YearFraction] = yearfrac(Date1, Date2, Basis)

Inputs:

- Date1 - [Nx1 or 1xN] vector containing values for Date 1 in either date string or serial date form
- Date2 - [Nx1 or 1xN] vector containing values for Date 2 in either date string or serial date form
- Basis - [Nx1 or 1xN] vector containing values that specify the Basis for each set of dates.

Comparing Matlab and Excel

	d_e	25 oct 96	27 jan 98
	d_t	31 dec 96	01 feb 99
Basis	Day Count	Matlab	Excel
0	actual/actual(default)	0.183561644	1.01369863
1	30/360 SIA	0.183333333	1.011111111
2	actual/360	0.186111111	1.027777778
3	actual/365	0.183561644	1.01369863
4	30/360 PSA	0.183333333	1.011111111
5	30/360 ISDA	0.183333333	1.011111111
6	30/360 European	0.180555556	1.011111111
7	actual/365 Japanese	0.183561644	1.01369863
8	actual/actual ISMA	0.183561644	1.01369863
9	actual/360 ISMA	0.186111111	1.027777778
10	actual/365 ISMA	0.183561644	1.01369863
11	30/360 ISMA	0.180555556	1.011111111

Table 3: Values in the third column have been produced with the Matlab function yearfrac('25 oct 96', '31 dec 96', [0:11]). Values in the fourth column have been produced with the Matlab function yearfrac('27 jan 98', '1 feb 99', [0:11]). Notice that the Matlab and Excel functions yearfrac do not provide always the same result (see the row labelledACT/ACT example)

Date rolling

Date rolling i

Source: `http://en.wikipedia.org/wiki/Accrued_interest`

- Date rolling comes into effect because many instruments can only pay out accrued interest on business days.
- Therefore, we have to *roll* the payment to a good business day.
- This often results in interest accruing for a slightly shorter or longer period.
- However, if interest is unadjusted, interest roll dates and payment dates can be different.
- Common date rolling conventions are of four different types:
    - Following business day
    - Modified following business day
    - Preceding business day
    - Modified previous business day

Date rolling ii

1. Following business day
    - The payment date is rolled to the next business day.
    - **Examples**: Start date 18-Aug-2011 (Thurs.), period 1 month: end date: 19-Sep-2011 (Monday).
2. Modified following business day.
    - This is the most used convention for interest rate derivatives.
    - The payment date is rolled to the next business day, unless doing so would cause the payment to be in the next calendar month, in which case the payment date is rolled to the previous business day.
    - **Examples**: Start date 30-Jun-2011, period 1 month: end date: 29-Jul-2011 (Friday). The following rule would lead to 1-Aug (Monday) which is in the next calendar month with respect to 30-Jul (Saturday).

Date rolling iii

3. Previous business day.
    - The payment date is rolled to the previous business day.
    - **Examples**: Start date 18-Aug-2011, period 1 month: end date: 16-Sep-2011 (Friday).
4. Modified previous business day.
    - The payment date is rolled to the previous business day, unless doing so would cause the payment to be in the previous calendar month, in which case the payment date is rolled to the next business day.
    - **Examples**: Start date 1-Apr-2011, period 1 month: end date: 2-May-2011 (Monday). The preceding rule would lead to 29-Apr (Friday) which is in the previous calendar month with respect to 1-May.

Date rolling iv

5. End of Month

   - Where the start date of a period is on the final business day of a particular calendar month, the end date is on the final business day of the end month (not necessarily the corresponding date in the end month).
   - Start date 28-Feb-2011, period 1 month: end date: 31-Mar-2011.
   - Start date 29-Apr-2011, period 1 month: end date: 31-May-2012. 30-Apr-2011 is a Saturday, so 29-Apr is the last business day of the month.
   - Start date 28-Feb-2012, period 1 month: end date: 28-Mar-2012. 2012 is a leap year and the 28th is not the last business day of the month!

6. IMM days: International Money Market or IMM days are the third Wednesday in March, June, September and December. They are used in the interest rate futures market.

Holidays

Holidays i

- The primary sources used depend on the currency and are as follows:
    - AUD: Reserve Bank of Australia (the holidays in New South Wales apply; e-mail contact: mailto:southersm@rba.gov.au)
    - CAD: Federal Bank of Canada (see `http://www.bank-banque-canada.ca`) and Canadian Bankers Association
    - HF: Swiss National Bank (see `http://www.snb.ch`)
    - KK: Danmarks Nationalbank (see `http://www.riksbank.com`)
    - EUR: TARGET holidays (as published by the European Central Bank)
    - GBP: Department of Trade and Industry (England and Wales)
    - JPY: Bank of Japan (see `http://www.boj.or.jp/en/about/holi.htm`; note that the dates of the invernal and autumnal equinoxes play a role in defining Japanese holidays)
    - NOK: Norges Bank (see `http://www.norges-bank.no`)
    - NZD: Federal Reserve Bank of New Zealand (see `http://www.rbnz.govt.nz/payment/ESAS/index.html`)
    - SEK: Sveriges Riksbank (see `http://www.nationalbanken.dk`)

Holidays ii

- USD: Federal Reserve Bank of New York [Federal Holidays], plus Good Friday [NYSE] (see http://www.ny.frb.org/bankinfo/services/frsholi.html and http://www.ny.frb.org/bankinfo/circular/11087.html)
- ZAR: Reserve Bank of South Africa (see http://www.resbank.co.za with email contact: mailto:info@resbank.co.za)

- Above information has been taken from: http://www.swx.com/download/trading/products/bonds/accrued_interest_en.pdf
- As a cross-check, the very comprehensive International Bank Holidays calendar published by the Banque Generale du Luxembourg is used (see http://www.bgl.lu).
- ISMA publishes holiday information (see http://www.isma.org).

Forward Prices and Rates

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MSc Mathematical Finance & Trading
MSc Quantitative Finance

SMM269 Fixed Income

Academic Year 2019-20

References

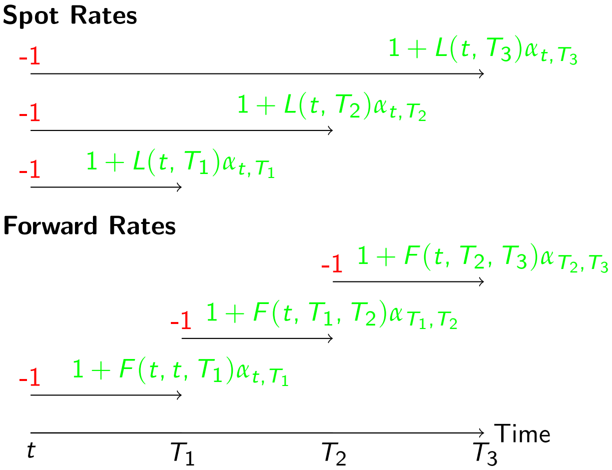
Useful Readings

- Veronesi P., (2010). Fixed Income Securities. Chapters: **5.1**, **5.2**, **5.3**.

Excel Files

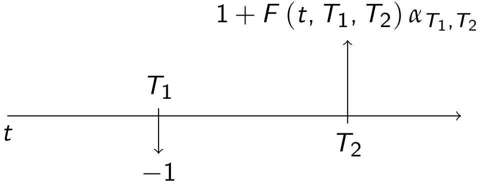
- FI_ForwardRates.xlsm

Outline


Spot Rates
-1
$1 + L(t, T_3)\alpha_{t,T_3}$
-1
$1 + L(t, T_2)\alpha_{t,T_2}$
-1
$1 + L(t, T_1)\alpha_{t,T_1}$
Forward Rates
-1
$1 + F(t, T_2, T_3)\alpha_{T_2,T_3}$
-1
$1 + F(t, T_1, T_2)\alpha_{T_1,T_2}$
-1
$1 + F(t, t, T_1)\alpha_{t,T_1}$
t
T_1
T_2
T_3
Time

Forward Deposit

1. On the Value Date t we fix the **forward rate** $F(t, T_1, T_2)$.
2. At time T_1 (the **Reset Date**) we make a deposit;
3. At time T_2 (the **Payment Date**) we receive back our deposit and interests
4. Interests are computed according to a given compounding convention. Here we use simple compounding.

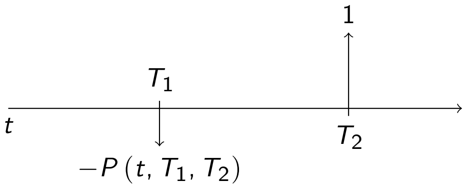


5. So the forward rate is the rate at which we could sign a contract today to borrow or lend between periods T_1 and T_2.
6. **Remark** Notice that if the reset date coincides with the Value Date, the forward rate becomes a spot rate

$$F(t, t, T_2) = L(t, T_2).$$

Forward Contract on a Zero-Coupon

1. On the Value Date t we fix the **forward price** $P(t, T_1, T_2)$.
2. At time T_1 we pay the forward price.
3. At time T_2 we receive one unit of currency.



Forward Price and Forward Rate

Forward Price and Forward Rate

- The `forward price` $P(t, T_1, T_2)$ is the price fixed in t and to be paid in T_1 to receive 1 USD in T_2.
- The `forward rate` $F(t, T_1, T_2)$ is the return of the forward operation.
- It can be computed according to different conventions (simple, annually, continuous).
- If we use simple compounding, then we have
  Direct formula

$$P(t, T_1, T_2) = \frac{1}{1 + F(t, T_1, T_2) \times \alpha_{T_1, T_2}}.$$

Inverse formula

$$F(t, T_1, T_2) = \frac{1}{\alpha_{T_1, T_2}} \left(\frac{1}{P(t, T_1, T_2)} - 1 \right).$$

Example (Forward Final value)

- The 6x9 FRA rate is 2.41%.
- How much do I receive in 9 months if I make a deposit (notional value=1000 Euro) in 6 months time?
- We have:

$$1000 \times \left(1 + 0.0241 \times \frac{90}{360}\right) = 1006.025 \quad \text{Euro.}$$

Example (Forward Present value)

- The simple forward rate for a 3x6 month deposit in USD is 1.29%.
- How much do I need to invest in 3 months time, if I want an amount of 1000USD to be available in 6 months time?
- Therefore:

$$\frac{1000}{\left(1 + 0.0129 \times \frac{90}{360}\right)} = 996.785 USD.$$

Question

- Can we establish a relationship between spot and forward prices/rates?

Joining Spot and Forward Rates

Spot Deposit up to T_2

-1 $\qquad$ $1 + L(t, T_2)\alpha_{t,T_2}$

Spot Deposit up to T_1 + Forward Deposit $T_1 \times T_2$

-1 $\qquad$ $1 + L(t, T_1)\alpha_{t,T_1}$ $\qquad$ $(1 + L(t, T_1)\alpha_{t,T_1}) \times (1 + F(t, T_1, T_2)\alpha_{T_1,T_2})$

$-(1 + L(t, T_1)\alpha_{t,T_1})$

t $\qquad$ T_1 $\qquad$ T_2 $\qquad$ Time

No arbitrage restriction

FV deposit up to T_2	=	FV deposit up to T_1 x FV forward deposit $T_1 \times T_2$

Table 1: Final Value (FV) in T_2 via two different strategies

The no arbitrage restriction is

$$1 + L(t, T_2)\alpha_{t,T_2} = (1 + L(t, T_1)\alpha_{t,T_1}) \times (1 + F(t, T_1, T_2)\alpha_{T_1,T_2})$$

and then we have a link between spot and forward rates

$P(t, T_2) = P(t, T_1) \times P(T_1, T_2)$

$$F(t, T_1, T_2) = \frac{1}{\alpha_{T_1,T_2}} \left(\frac{1 + L(t, T_2) \times \alpha_{t,T_2}}{1 + L(t, T_1) \times \alpha_{t,T_1}} - 1 \right). \quad (1)$$

simple compounding convention

Forward and Spot Prices

Let us obtain a no-arbitrage relationship linking forward prices and spot prices.

- Let us consider the following strategy:

	t	T_1	T_2
buy a T_2 zcb	$-P(t, T_2)$		1
sell k T_1 zcb	$kP(t, T_1)$	$-k$	
cash flows	$kP(t, T_1) - P(t, T_2)$	$-k$	1

Forward and Spot Prices

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	t	T_1	T_2
buy a T_2 zcb	$-P(t, T_2)$		1
sell k T_1 zcb	$kP(t, T_1)$	$-k$	
cash flows	$kP(t, T_1) - P(t, T_2)$	$-k$	1

- Choose k such that the cash flows at t are 0:

$$kP(t, T_1) - P(t, T_2) = 0,$$

i.e., we have

$$k = \frac{P(t, T_2)}{P(t, T_1)}.$$

Forward and Spot Prices

Let us obtain a no-arbitrage relationship linking forward prices and spot prices.

- Let us consider the following strategy:

	t	T_1	T_2
buy a T_2 zcb	$-P(t, T_2)$		1
sell k T_1 zcb	$kP(t, T_1)$	$-k$	
cash flows	$kP(t, T_1) - P(t, T_2)$	$-k$	1

- Choose k such that the cash flows at t are 0:

$$kP(t, T_1) - P(t, T_2) = 0,$$

i.e., we have

$$k = \frac{P(t, T_2)}{P(t, T_1)}.$$

- The cash flows become

	t	T_1	T_2
cash flows	0	$-\frac{P(t,T_2)}{P(t,T_1)}$	1

Forward Prices

- The strategy consists of:
  1. zero-investment at time t;
  2. a payment of amount $\frac{P(t,T_2)}{P(t,T_1)}$ at time T_1;
  3. receive a unit amount at time T_2.

	t	T_1	T_2
cash flows	0	$-\frac{P(t,T_2)}{P(t,T_1)}$	1

- In a forward contract the cash flows are

	t	T_1	T_2
cash flows	0	$-P(t, T_1, T_2)$	1

- No arbitrage implies that

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}. \quad (2)$$

No Arbitrage forward price and rate

No arbitrage Forward Price

No arbitrage says that the forward price is related to the zcb spot prices trough

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)} = \frac{1}{\alpha_{T_1, T_2}} \left(\frac{1 + L(t, T_1)\alpha_{t, T_1}}{1 + L(t, T_2)\alpha_{t, T_2}} - 1 \right).$$

No-Arbitrage Forward Rate

The simple forward rate is related to spot zcb prices according to the formula

$$F(t, T_1, T_2) = \frac{1}{\alpha_{T_1, T_2}} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right) = \frac{1}{\alpha_{T_1, T_2}} \left(\frac{1 + L(t, T_2)\alpha_{t, T_2}}{1 + L(t, T_1)\alpha_{t, T_1}} - 1 \right).$$

Remark: The above formula assumes simple compounding. If we use different compounding rules, the forward rate will be computed in a different way.

Question

- The 6m USD LIBOR (182 days) is 1.25%.
- The 7m USD LIBOR (209 days) is 1.35%.

What is the 6x7 forward rate?

Answer

- The 6m USD LIBOR (182 days) is 1.25 %. Therefore the 6m spot zcb price is

$$\frac{1}{1+0.0125\times\frac{182}{360}}=0.993720.$$

- The 7m USD LIBOR (209 days) is 1.35 %. Therefore the 7m spot zcb price is

$$\frac{1}{1+0.0135\times\frac{209}{360}}=0.99222.$$

- The 6x7 forward rate is

$$\frac{1}{\frac{209-182}{360}}\times\left(\frac{0.993720}{0.99222}-1\right)=\frac{1}{\frac{209-182}{360}}\times 0.00150=2.0114\%.$$

- Or using (1) at page 12, we have

$$F(0,6m,7m)=\frac{1}{\frac{7-6}{12}}\left(\frac{1+1.35\%\frac{7}{12}}{1+1.25\%\frac{6}{12}}-1\right)=2.0114\%. \qquad (3)$$

How many forward curves? I

- To a given term structure of spot rates we can associate forward curves with different tenors.
- The most important are the ones with O/N, 1m, 3m, 6m and 12m tenors.
  - If we have a 3m tenor forward curve, we are considering: F(0,0,3m); F(0,3m,6m); F(0,6m,9m); F(0,9m,12m); etc.
  - If we have a 6m tenor forward curve, we are considering: F(0,0,6m); F(0,6m,12m); F(0,12m,18m); F(0,18m,24m); etc.
  - If we have a 12m tenor forward curve, we are considering: F(0,0,12m); F(0,12m,24m); F(0,24m,36m); F(0,36m,48m); etc.

How many forward curves? II

Monthly Tenor $F(t, T_i, T_i + 1m)$

0×1 .... 23×24

Quarterly Tenor $F(t, T_i, T_i + 3m)$

0×3 3×6 6×9 9×12 12×15 15×18 18×21 21×24

Semi-Annual Tenor $F(t, T_i, T_i + 6m)$

0×6 6×12 12×18 18×24

Annual Tenor $F(t, T_i, T_i + 12m)$

0×12 12×24

Figure 1: Illustration of different tenor structures

T_i	$L(t,T_i)$%	$P(t,T_i)$	$P(t,T_i,T_{i+1})$	$F(t,T_i,T_{i+1})$%
1	2.095	0.9983	0.9983	2.0950
2	2.110	0.9965	0.9982	2.1213
3	2.154	0.9946	0.9981	2.2341
4	2.172	0.9928	0.9982	2.2141
5	2.190	0.9910	0.9981	2.2457
6	2.208	0.9891	0.9981	2.2772
7	2.236	0.9871	0.9980	2.3777
8	2.267	0.9851	0.9980	2.4520
9	2.299	0.9830	0.9979	2.5170
10	2.330	0.9810	0.9979	2.5648
11	2.361	0.9788	0.9978	2.6201
12	2.384	0.9767	0.9979	2.5811

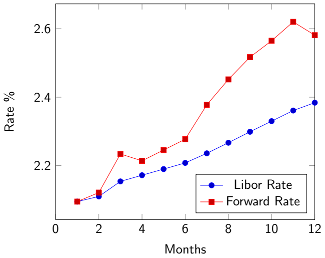


Figure 2: Left Panel: Term Structure of LIBOR rates, discount factors, forward discount factors and forward rates. The forward price 2x3 is computed by taking the ratio of the 2 month and 3 months discount factor, i.e. $0.9946/0.9965 = 0.9981$. The 2x3 forward rate is then computed out of the forward price, e.g. $(1/0.9981 - 1)/(1/12) = 2.2341\%$. **Right Panel:** LIBOR curve (different tenors) and Forward Curve (1 month tenor).

Term	Discount Factor	Fwd 3m	Fwd 6m	Fwd 12m
0.00	100.0000%			
0.25	100.0771%	-0.3083%		
0.50	100.1227%	-0.1821%	-0.2451%	
0.75	100.1374%	-0.0585%		
1.00	100.1217%	0.0625%	0.0020%	-0.1216%
1.25	100.0764%	0.1811%		
1.50	100.0021%	0.2972%	0.2392%	
1.75	99.8995%	0.4108%		
2.00	99.7693%	0.5221%	0.4667%	0.3532%
2.25	99.6122%	0.6310%		
2.50	99.4288%	0.7375%	0.6848%	
2.75	99.2201%	0.8417%		
3.00	98.9866%	0.9436%	0.8936%	0.7908%

Table 2: Term structure of simple forward rates with different tenors.

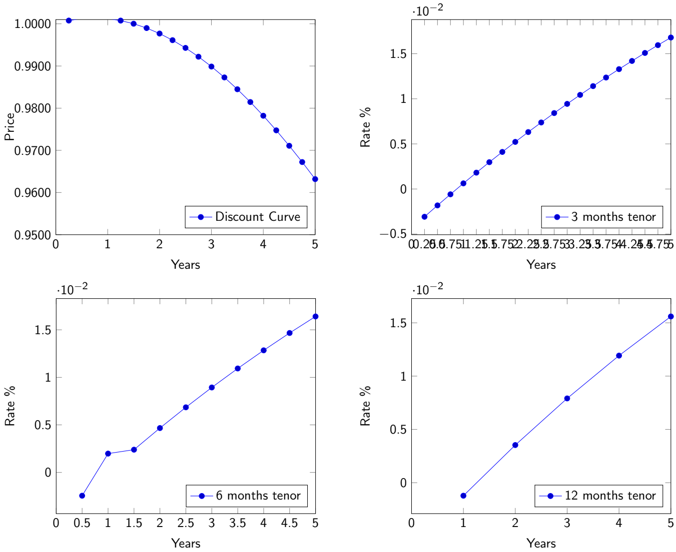


Table 3: Top Left: Discount Curve. **Top Right**: Fwd Curve 3m tenor. **Bottom Right**: Fwd Curve 6m tenor. **Bottom Left**: Fwd Curve 12m tenor

Where will be the rates in the future? I

Where will be the rates in 1 year?
and in 2 years?

Fact

- *Forward rates provide so called market risk-neutral expectations of future rates.*
- *So, if we are interested in a possible scenario of the term structure in 1 year, we can compute forward rates starting in 1 year and with different horizons.*
- *If the market forecasts will be realized those should be the future values of the spot rates.*
- *Notice that in a deterministic world, interest rates can change over time, but they move according to the current forward curve.*
- *So if the 6m LIBOR is 1.25%, the 9m LIBOR is 1.35% and the 6x9 forward rate is 2.0114%, in a deterministic world the 3m LIBOR that we will observe in 6 monts time should be equal to 2.0114%.*

Where will be the rates in the future? II

Table 4: Term Structure Scenarios according to the forward curve: **99.65=99.77/100.12**. Rates computed according to the continuously compounded convention.

Term	Spot Today	DF	DF in 1 Year	Spot in 1 yr	DF in 2 yrs	Spot in 2 yrs
0.00	-0.37%	100.00%	100.00%		100.00%	
0.25	-0.31%	100.08%	99.95%	0.18%	99.84%	0.63%
0.50	-0.25%	100.12%	99.88%	0.24%	99.66%	0.68%
0.75	-0.18%	100.14%	99.78%	0.30%	99.45%	0.74%
1.00	-0.12%	100.12%	99.65%	0.35%	99.22%	0.79%
1.25	-0.06%	100.08%	99.49%	0.41%	98.96%	0.84%
1.50	0.00%	100.00%	99.31%	0.46%	98.68%	0.89%
1.75	0.06%	99.90%	99.10%	0.52%	98.37%	0.94%
2.00	0.12%	99.77%	98.87%	0.57%	98.05%	0.99%
2.25	0.17%	99.61%	98.61%	0.62%	97.70%	1.03%
2.50	0.23%	99.43%	98.33%	0.67%	97.33%	1.08%
2.75	0.28%	99.22%	98.03%	0.73%	96.95%	1.13%
3.00	0.34%	98.99%	97.70%	0.78%	96.54%	1.17%

Where will be the rates in the future? III

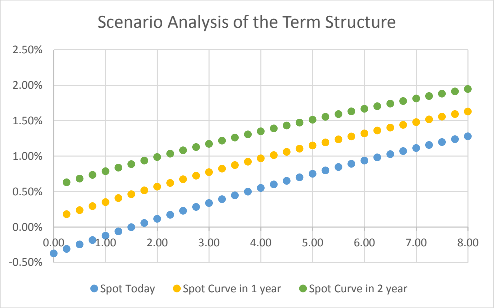


Figure 3: We use the forward curve to build future scenarios of the Term Structure of (continuously compounded) Spot Rates.

Case Study
Estimating the cash flows of a FRN

Estimating the cash flows of a FRN I

- We are interested in estimating the cash flows of a FRN
- The FRN has quarterly coupons according to the formula

$$3\text{m US Libor} + 100 \text{ bp}$$

- The FRN expires in 8 months.
- The current coupon is 2.23%.

Reset Date (m)	Payment Date (m)	Cash Flow
-1	2	$2.23\% x 0.25$
2	5	$(L(2m, 5m) + 1\%) x 0.25$
5	8	$(L(5m, 8m) + 1\%) x 0.25 + 1$

Table 5: Cash Flow schedule

Estimating the cash flows of a FRN II

- We can estimate the unknown coupon rates due in 5 and 8 months.

a. For the coupon due in 2 months we need the current coupon rate.

b. The coupon that is paid in 5 months resets in 2 months. We need the 2x5 forward rate.

c. The coupon that is paid in 8 months resets in 5 months. We need the 5x8 forward rate.

- Linearly interpolating reported LIBOR rates for terms of 5m, and 8m we can compute the relevant discount factors.
- Then we can compute the 2x5 and 5x8 forward rates.
- Let us see the LIBOR quotes on the trade date (6th Feb. 2018).

Estimating the cash flows of a FRN III

We have the following term structure of LIBOR rates

Table 6: US Libor rates on 6th February 2018

Tenor	Rate
overnight	1.43875 %
1 week	1.46875 %
2 weeks	-
1 month	1.57926 %
2 months	1.67149 %
3 months	1.79070 %
6 months	1.99188 %
12 months	2.27825 %

Estimating the cash flows of a FRN IV

Table 7: We need 5m and 8m LIBOR rates. Interpolation of quoted rates to reconduct to the relevant dates

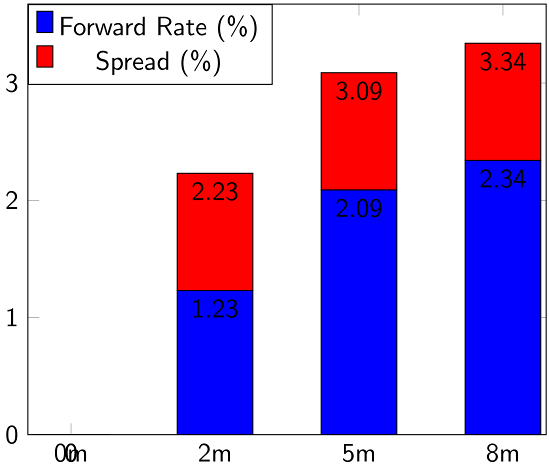
							Interpolated	
Time	Adjacent Rates		Tenors		Weights		Rate	DF
2							1.6715%	0.99722
5	1.7907%	1.9919%	3	6	33.33%	66.67%	1.9248%	0.99204
8	1.9919%	2.2783%	6	12	66.67%	33.33%	2.0873%	0.98628

Estimating the cash flows of a FRN V

Table 8: Computing expected coupons of the FRN, replacing the unknown LIBOR rate in the coupon formula by the corresponding forward rate

Term	DF	Tenor	Fwd Rate	Spread	Cpn Rate	Cash Flow
0	1					
2	0.99722	0.25			2.2300%	2.2300% $\times$ 0.25
5	0.99204	0.25	2.0879%	1%	3.0879%	3.0879% $\times$ 0.25
8	0.98628	0.25	2.3394%	1%	3.3394%	3.3394% $\times$ 0.25

Estimated coupons (%) of the FRN



Case Study:
Forward Price of a Coupon Bond

Buying Forward a Coupon Bond

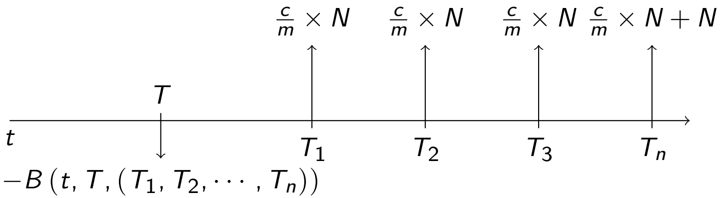


Figure 4: Cash flows on a forward coupon bond

Forward Price of a Coupon Bond

Example (Forward Price of a Coupon bond)

- We have to compute the forward price on a bond. This price is fixed today but paid forward.
- We buy the bond in 6 months.
- The bond expires in 1 year and pays quarterly coupons.
- The notional coupon of the bond is 4%.

Term	0	0.25	0.5	0.75	1
Event			Buy Forward	Cpn	Cpn+FV
Cash Flows				1	101

- **Notice that we have only to consider cash flows occurring after the option expiry, i.e. in 9 and 12 months.!**

Example ((...continued))

- Let us compute the forward price of the bond.
- This a portfolio of forward zero-coupon bonds.
- Determine the forward price of the zcbs, 6x9 and 6x12 and sum up.
- The term structure of discount factors is given in the following Table

Time	0.25	0.5	0.75	1
DF	0.99	0.98	0.97	0.96

- In 9m the coupon cash flow is

$$0.04 \times 0.25 \times 100 = 1.$$

- In 12m the cash flow is

$$0.04 \times 0.25 \times 100 + 100 = 101.$$

- The forward price of the bond for delivery in 6 months is

$$1 \times \frac{0.97}{0.98} + 101 \times \frac{0.96}{0.98} = 0.9898 = 98.9388 = 99.9286.$$

Question

$$1 \times \frac{0.98}{0.99} + 1 \times \frac{0.97}{0.99} + 101 \times \frac{0.96}{0.99}$$

Determine the 3m forward price of the coupon bond in the last Case Study.

Answer

Conclusions

- We have introduced forward contracts.
- How to determine the rate on a forward deposit.
- The forward term structure.
- How to use forward rates.
- How to use forward prices.

Instantaneous forward rate

- Given the forward rate $F(t, T, T+\Delta)$ we let $\Delta \to 0$.
- We can define the so called instantaneous forward rate:

$$\begin{aligned} f(t, T) &= \lim_{\Delta t \to 0} F(t, T, T+\Delta t) \\ &= -\lim_{\Delta t \to 0} \frac{\ln P(t, T+\Delta t) - \ln P(t, T)}{\Delta t} \\ &= -\frac{\partial \ln P(t, T)}{\partial T}. \end{aligned}$$

- $f(t, T)$ is then the return on a forward contract stipulated at time t, with starting date T and instantaneous expiry in $T + \Delta t$.
- Viceversa, we have:

$$P(t, T) = e^{-\int_t^T f(t,s) ds}.$$

The term structure of instantaneous forward rates

- The instantaneous forward term structure is intended as the plot at time t of $f(t, T)$ starting at date T with infinitesimal maturity.
- Note that for this curve, it is the starting date of the forward deposit T that changes, not the maturity of the instantaneous forward rate.
- The tenor of the forward deposit is infinitesimal.
- Note that the forward term structure is a implied curve: it can be constructed once we have $P(t, T), \forall T > t$.
- In general, $f(t, T)$ is not a traded quantity.

$f(t,T)$ and $R(t,T)$

- The knowledge of all instantaneous forward rates for all $T \geq t$ at a given time t, allows the determination of different quantities.
- From the relationship between $P(t,T)$ and $R(t,T)$, we have:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} = \frac{\int_t^T f(t,s)ds}{T-t},$$

i.e. the spot rate is an average of forward rates.

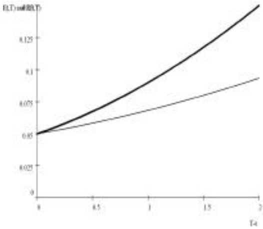
- Viceversa:

$$f(t,T) = \frac{\partial (T-t) R(t,T)}{\partial T} = R(t,T) + (T-t)\frac{\partial R(t,T)}{\partial T}.$$

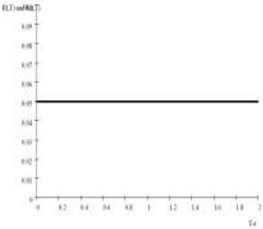
i.e. forward rates are related to the level and slope of the term structure of spot rates.

Spot and forward rates

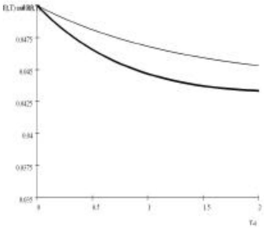
- Given the relationship between $f\left(t,T\right)$ and spot rates $R\left(t,T\right)$, if we plot the term structure of forward rates we have:
  1. $f\left(t,T\right) \geq R\left(t,T\right)$ if $R\left(t,T\right)$ is increasing in $T-t$;
  2. $f\left(t,T\right) = R\left(t,T\right)$ if $R\left(t,T\right)$ is flat;
  3. $f\left(t,T\right) \leq R\left(t,T\right)$ if $R\left(t,T\right)$ is decreasing in $T-t$



1. $f\left(t,T\right) \geq R\left(t,T\right)$



2. $f\left(t,T\right) = R\left(t,T\right)$



3. $f\left(t,T\right) \leq R\left(t,T\right)$

The instantaneous interest rate $r(t)$ I

The instantaneous interest rate, or `short rate`, is the return on a spot deposit of infinitesimal length.

- Let us consider $r(t)$. By definition:

$$r(t) = \lim_{T \to t} R(t, T) = \lim_{T \to t} \frac{\int_t^T f(t,s) ds}{T - t} = f(t, t)$$

i.e. the instantaneous interest rate is a particular forward rate.

- The knowledge of $r(t)$ does not say anything about $f(t, T)$, $T > t$.
- Only if we known the future path of $r(.)$, we can recover $f(t, T)$:

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T} = \frac{\partial \int_t^T r(s)\, ds}{\partial T} = r(T)$$

- Then, `under a deterministic evolution`, the instantaneous interest rate at time T is equal to the current instantaneous forward rate $f(t, T)$.

The instantaneous interest rate $r(t)$ II

- Remarks:
  1. It is a convenient quantity to use for modelling purposes: the math is simpler. For this reason, the first term structure models (Vasicek and CIR) were short rate models, i.e. models assigning the dynamics of $r(t)$.
  2. Note that $r(t)$ does not depend on the maturity T any longer.
  3. $r(t)$ represents just a point on the term structure of spot rates: it is the intercept on the vertical axis.
  4. $r(t)$ does not exist as traded quantity in the market.

$r(t)$ and $P(t, T)$

- In general, we cannot recover the whole zero-rate curve by knowing the short rate at time t only.
- Indeed, given the discount curve we can obtain $r(t)$ according to the previous formula
$$P(t, T), \forall T \quad \Rightarrow \quad r(t).$$
- Viceversa, given the short rate we cannot recover the discount curve
$$r(t) \quad \not\Rightarrow \quad P(t, T).$$
- However, assuming a future deterministic path for $r(.)$, i.e. if we know the entire future path $r(s)$, $t \leq s \leq T$, then
$$P(t, T) = e^{-\int_t^T r(s)ds}.$$
- In general, outside the deterministic world, this relationship is not true.
- In order to understand the relatioship between P and r, we need to remember some basic fact on no-arbitrage pricing. In particular, we need to remember the concept of `money market account`, `risk-neutral expectation` and `martingale`.

Bootstrapping Term Structures

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SMM269 Fixed Income

Academic Year 2019-20

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- Ametrano, Ferdinando M. and Bianchetti, Marco, Everything You Always Wanted to Know About Multiple Interest Rate Curve Bootstrapping but Were Afraid to Ask (April 2, 2013). Available at SSRN: https://ssrn.com/abstract=2219548.

Excel Files

- FI_Bootstrapping.xlsm

Outline

Introduction I

- In the valuation of fixed income securities, it is not the Treasury yield curve (yield to maturity versus maturity) that is used as the basis for determining the appropriate discount rate for computing the present value of cash flows but the Treasury spot rates.
- The Treasury spot rates are derived from the Treasury bond prices using the bootstrapping process.
- Similarly, it is not the swap curve that is used for discounting cash flows when the swap curve is the benchmark but the spot rates. The spot rates are derived from the swap curve in exactly the same way, using the bootstrapping methodology.
- The resulting spot rate curve is called the LIBOR spot rate curve.

Introduction II

- Moreover, a forward rate curve can be derived from the spot rate curve. The same thing is done in the swap market.
- The forward rate curve that is derived is called the LIBOR forward rate curve.
- In the United States it is common to use the Treasury spot rate curve for purposes of valuation.
- In other countries, either a government spot rate curve is used (if a liquid market for the securities exists) or the swap curve is used (or as explained shortly, the LIBOR curve).

Bootstrapping

- The boostrapping approach consists of recovering discount factors from prices of coupon bearing bonds.
- The bootstrapping approach requires having at least one zero-coupon bond.
- Given this bond's rate, a coupon bond with the next highest maturity is used to obtain an implied spot rate
- Another coupon bond with the next highest maturity is then used to find the next spot rates, and so on.

Bootstrapping using Coupon Bonds I

- Let t be the Value Date.
- Let us suppose we have n coupon bonds, with coupons c_i and maturities T_i and prices $B_i = B(c_i, t; T_i)$.
- Assume that available bonds have maturities at regular intervals (e.g. 6m, 12m, 18m etc or 12m, 24m, 36m etc.) and let us define $\alpha_{i-1,i}$ the appropriate coupon day count fraction.

Bond	Gross Price	Cpn Rate	Payment Dates & Cash Flows 1	2	$\cdots$	n
1	B_1	c_1	$c_1\alpha_{0,1}$			
2	B_2	c_2	$c_2\alpha_{0,1}$	$c_2\alpha_{0,1}+1$		
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
n	B_n	c_n	$c_n\alpha_{0,1}$	$c_n\alpha_{1,2}$		$c_n\alpha_{n-1,n}+1$

in swap market
\# of bonds = \# of maturities
→ unique solution

Bootstrapping using Coupon Bonds II

- The bootstrap procedure allows to obtain the zero-coupon discount factors according to the recursive procedure

$$P(t, T_1) = \frac{B_1}{1 + c_1 \times \alpha_{0,1}},$$

and then for $i > 1$

$$P(t, T_n) = \frac{B_n - c_n \times A_{n-1}}{1 + c_n \times \alpha_{n-1,n}},$$

where

$$A_1 = P(t, T_1)\alpha_{0,1},$$

and then we have the recursion

$$A_n = \sum_{i=1}^{n} P(t, T_i)\alpha_{i-1,i} = A_{n-1} + P(t, T_n)\alpha_{n-1,n}.$$

Limits

- It is not always possible to find bond data at equally spaced intervals.
- In general the period to the next coupon date is different from the coupon frequency (except in the exceptional case of being at a coupon date).
- These problems limit the use of bonds in the bootstrapping iterative procedure.
- Instead, bootrstrapping is the standard procedure in the swap market.
- In this markets quotations of par bonds at equally spaced intervals are promptly available, althought some interpolation is still required to fill the missing maturities (eg moving from 10 to 12 years).

Example (1. Collecting Market Information)

Excel file: FI_Bootstrappings, Sheet: Ex Bootstrapping Bond

We have 3 bonds with annual coupons and expiring in 1, 2 and 3 years. Quotes are as in Table 1.

Maturity	Annual Coupon	Principal	Price
1	3%	100	98
2	4%	100	101
3	5%	100	103

Table 1: Market Quotations

We can build a cash-flow matrix, as in Table 2.

Dates	0	1	2	3
Bond 1	-98	103		
Bond 2	-101	4	104	
Bond 3	-103	5	5	105

Table 2: Cash Flow Matrix

Example (2. Bootstrapping: recovering implied discount factors)

For each bond, we can write the pricing equation: i.e. we set the gross price to be equal to the present value of the bond cash flows.

1. Using the first bond, we have

$$98 = 103 \times P(0,1),$$

and therefore

$$P(0,1) = 0.951456, \quad A(1) = 0.951456 \times (1-0) = 0.951456.$$

2. Using the second bond, we have

$$101 = 4 \times P(0,1) + 104 \times P(0,2),$$

and therefore

$$P(0,2) = \frac{101 - 4 \times 0.951456}{104} = 0.93456,$$

and

$$A(2) = A(1) + 0.93456 \times (2-1) = 0.95146 + .93456 = 1.88602.$$

Example ((..ctd))

1. Using the third bond, we have

$$103 = 5 \times P(0,1) + 5 \times P(0,2) + 105 \times P(0,3),$$

and therefore

$$P(0,3) = \frac{103 - 5 \times (0.951456 + 0.93456)}{105} = \frac{103 - 5 \times \mathbf{1.88602}}{105} = 0.89114,$$

and

$$A(3) = A(2) + 0.89114 \times (3 - 2) = 2.77716.$$

2. Therefore, we have the following term structure of discount factors implied by market prices of coupon bonds

Time (years) T	1	2	3
$P(0, T)$	0.951456	0.93456	0.89114
$A(T)$	0.951456	1.88602	2.77716

Table 3: Bootstrapped discount curve

Example ((..ctd))

1. We can check the correctness of our calculations by repricing the three bonds using the bootstrapped discount curve and the annuity.
2. Bond 1
$$B_1 = 103 \times 0.951456 = 98.$$
3. Bond 2
$$B_2 = 4 \times A(2) + 100 \times P(2) = 7.54408 + 93.456 = 101.$$
4. Bond 3
$$B_3 = 5 \times A(3) + 100 \times P(3) = 13.8858 = 89.114 = 103.$$
5. So we have repriced correctly all the bonds used in the bootstrapping procedure.

Reasons for Increased Use of Swap Curve

- Investors and issuers use the swap market for hedging and arbitrage purposes, and the swap curve as a benchmark for evaluating performance of fixed income securities and the pricing of fixed income securities.
- Since the swap curve is effectively the LIBOR curve and investors borrow based on LIBOR, the swap curve is more useful to funded investors than a government yield curve.
- The increased application of the swap curve for these activities is due to its advantages over using the government bond yield curve as a benchmark.
- The drawback of the swap curve relative to the government bond yield curve could be poorer liquidity. In such instances, the swap rates would reflect a liquidity premium.
- Fortunately, liquidity is not an issue in many countries as the swap market has become highly liquid, with narrow bid-ask spreads for a wide range of swap maturities. In some countries swaps may offer better liquidity than that country's government bond market.

Advantages of the swap curve over a government bond yield curve I

- There is almost no government regulation (e.g. tax aspects) of the swap market, that makes swap rates across different markets more comparable.
  - In some countries, there are some sovereign issues that offer various tax benefits to investors and, as a result, for global investors it makes comparative analysis of government rates across countries difficult because some market yields do not reflect their true yield.
- The supply of swaps depends only on the number of counterparties that are seeking or are willing to enter into a swap transaction at any given time. Since there is no underlying government bond, there can be no effect of market technical factors that may result in the yield for a government bond issue being less than its true yield.
- Comparisons across countries of government yield curves is difficult because of the differences in sovereign credit risk. Sovereign risk is not present in the swap curve because, as noted earlier, the swap curve is viewed as an inter-bank yield curve or AA yield curve.

Advantages of the swap curve over a government bond yield curve II

- There are more maturity points available to construct a swap curve than a government bond yield curve.
  - More specifically, what is quoted in the swap market are swap rates for 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, and 30 year maturities. Thus, in the swap market there are 10 market interest rates with a maturity of 2 years and greater.
  - In contrast, in the U.S. Treasury market, for example, there are only three market interest rates for on-the-run Treasuries with a maturity of 2 years or greater (2, 5, and 10 years) and one of the rates, the 10-year rate, may not be a good benchmark because it is often on special in the repo market.
  - Moreover, because the U.S. Treasury has ceased the issuance of 30-year bonds, there is no 30-year yield available.

Swap Rates and Par Bonds

- In the swap market we find quotations of the so called par coupon rates, i.e. the coupon rate so that the underlying bond quotes at par.
- This par rate is called swap rate.
- The swap rate S_n of maturity T_n solves the equation

$$1 = S_n \sum_{i=1}^{n} \alpha_{i-1,i} P(t, T_i) + P(t, T_n).$$

- In the market we find quotes for S_n over a very wide spectrum of maturities.

Forward rate ≈ slope of discount curve

INTEREST RATES - SWAPS

	Euro-€		£ Stig.		SwFr		US $		Yen	
Dec 31	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	
1 year	0.14	0.18	0.63	0.66	-0.14	-0.08	0.42	0.45	0.11	0.17
2 year	0.16	0.20	0.91	0.95	-0.18	-0.10	0.86	0.89	0.11	0.17
3 year	0.20	0.24	1.11	1.15	-0.14	-0.06	1.26	1.29	0.13	0.19
4 year	0.26	0.30	1.28	1.33	-0.07	0.01	1.55	1.58	0.15	0.21
5 year	0.34	0.38	1.42	1.47	0.02	0.10	1.75	1.78	0.19	0.25
6 year	0.42	0.46	1.53	1.58	0.11	0.19	1.90	1.93	0.24	0.30
7 year	0.51	0.55	1.62	1.67	0.21	0.29	2.02	2.05	0.30	0.36
8 year	0.60	0.64	1.69	1.74	0.30	0.38	2.11	2.14	0.36	0.42
9 year	0.70	0.74	1.76	1.81	0.39	0.47	2.19	2.22	0.42	0.48
10 year	0.79	0.83	1.82	1.87	0.47	0.55	2.26	2.29	0.49	0.55
12 year	0.95	0.99	1.91	1.98	0.59	0.69	2.37	2.40	0.61	0.69
15 year	1.12	1.16	2.02	2.11	0.75	0.85	2.48	2.51	0.82	0.90
20 year	1.30	1.34	2.12	2.25	0.95	1.05	2.59	2.62	1.09	1.17
25 year	1.39	1.43	2.15	2.28	1.06	1.16	2.64	2.67	1.22	1.30
30 year	1.44	1.48	2.17	2.30	1.11	1.21	2.67	2.70	1.29	1.37

Bid and Ask rates as of close of London business. £ and Yen quoted on a semi-annual actual/365 basis against 6 month Libor with the exception of the 1Year GBP rate which is quoted annual actual against 3M Libor. Euro/Swiss Franc quoted on an annual bond 30/360 basis against 6 month Euribor/Libor. Source: ICAP plc.

Figure 1: IRS quotes on 31 Dec 2014. Source: Financial Times Data Archive, `http://markets.ft.com/research/Markets/Data-Archive`

Swap rates and market conventions

- Different payment frequencies, compounding frequencies and day count conventions are applicable to each currency-specific interest rate type.

Currency	EURO	JPY	USD	GBP	CHF
Index	EURIBOR or LIBOR	LIBOR or TIBOR	LIBOR	LIBOR	LIBOR
Fixed Leg					
Payment freq.	A	S/A	S/A	A for 1yr then S/A	A
Day Count Convention	$\frac{30}{360}$	$\frac{ACT}{365}$	$\frac{30}{360}$	$\frac{ACT}{365}$	$\frac{30}{360}$
Floating Leg					
Payment freq.	3m for 1yr then 6m	6m	3m	6m	3m for 1yr then 6m
Day Count Convention	$\frac{ACT}{360}$	$\frac{ACT}{360}$	$\frac{ACT}{360}$	$\frac{ACT}{365}$	$\frac{ACT}{360}$
Business Days	Target	Tokyo	New York	London	Zurich
Roll Day			modified following		

Table 4: Quotation Basis for Interest Rate Swaps

Table 5: Cash Flows Array of Par Swap Rates (Valid for EUR)

	Term															
Swap Rate (%)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.14	-100	100.14														
0.16	-100	0.16	100.16													
0.2	-100	0.2	0.2	100.2												
0.26	-100	0.26	0.26	0.26	100.26											
0.34	-100	0.34	0.34	0.34	0.34	100.34										
0.42	-100	0.42	0.42	0.42	0.42	0.42	100.42									
0.51	-100	0.51	0.51	0.51	0.51	0.51	0.51	100.51								
0.6	-100	0.6	0.6	0.6	0.6	0.6	0.6	0.6	100.6							
0.7	-100	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	100.7						
0.79	-100	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79	100.79					
?	Quotes not available															
0.95	-100	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.95	100.95			

Table 6: EURO MARKET: Swap Rate is quoted on an annual basis. Fixed leg swap payments are annual. The effective payment is computed on a semi-annual basis. Blue line signals missing quotations.

Table 7: Cash Flows Array of Par Swap Rates (Valid for USD, GBP and YEN)

	Term						
Swap Rate (%)	0	0.5	1	1.5	2	2.5	3
0.42	-100	0.21	100.21				
?	Quotes not available						
0.86	-100	0.43	0.43	0.43	100.43		
?	Quotes not available						
1.26	-100	0.63	0.63	0.63	0.63	0.63	100.63

Table 8: US MARKET: Swap Rate is quoted on an annual basis. Fixed leg swap payments are **semi-annual**. The effective payment is computed on a semi-annual basis. For example, $0.21 = 0.42 \times 0.5$, $0.43 = 0.86 \times 0.5$, etc.. Blue line signals missing quotations.

Example (Step 1. Bootstrapping Swap Rate)

Accompanying Excel file: FI_Bootstrappings, Sheet: Ex Bootstrapping Swap

Table 9: Bootstrapping Discount Factors from Swap Rates

Maturity	Swap Rate	Principal	Price
1	3%	100	100
2	4%	100	100
3	5%	100	100
Cash Flow Matrix			
0	1	2	3
-100	103		
-100	4	104	
-100	5	5	105

Example (Step 2. Recovering Discount Factors)

We have

T	$P(0, T)$	Annuity
1	0.970874	0.970874
2	0.924197	1.895071
3	0.862139	2.757210

Table 10: Bootstrapped Discount Curve

Indeed, we can recover market prices

- 1 year maturity
$$103 \times 0.970874 = 100.$$
- 2 years maturity
$$4 \times 1.895971 + 100 \times 0.924197 = 100.$$
- 3 years maturity, i.e. $5 \times 2.757210 + 100 \times 0.862139 = 100.$

Case Study

- Implement the bootstrapping procedure for EUR Swaps.
- How do you deal with missing rates?
- Compute the term structure of spot rates
- Compute the term structure of (simple) forward rates

$$F(t, T_{i-1}, T_i) = \frac{1}{\alpha_{T_{i-1},T_i}} \left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right)$$

- Discuss the issues you encounter
- Then implement the bootstrapping procedure for for GBP, USD and Yen.

This example can be found in the Excel File **FI_Bootstrapping.xlsm**, Sheet **BootstrappingSwapLinear**.

Table 11: Input: EURO Market Swap Rates

Term		Bid	Ask
1	year	0.14	0.18
2	year	0.16	0.2
3	year	0.2	0.24
4	year	0.26	0.3
5	year	0.34	0.38
6	year	0.42	0.46
7	year	0.51	0.55
8	year	0.6	0.64
9	year	0.7	0.74
10	year	0.79	0.83
12	year	0.95	0.99
15	year	1.12	1.16
20	year	1.3	1.34
25	year	1.39	1.43
30	year	1.44	1.48

Table 12: The Output: bootstrapped term structures of discount factors, forward and spot rates. Gray cells refer to linearly interpolated swap rates.

Term	Rate	$\alpha_{i-1,i}$	$P(0, T_i)$	Annuity	Fwd Rate	Spot Rate
1	0.66	1	99.34%	99.34%	0.6600%	0.6578%
2	0.95	1	98.12%	197.47%	1.2436%	0.9469%
3	1.15	1	96.62%	294.09%	1.5588%	1.1468%
4	1.33	1	94.83%	388.91%	1.8882%	1.3278%
5	1.47	1	92.92%	481.83%	2.0560%	1.4693%
6	1.58	1	90.95%	572.78%	2.1628%	1.5810%
7	1.67	1	88.95%	661.73%	2.2495%	1.6729%
8	1.74	1	86.97%	748.70%	2.2726%	1.7447%
9	1.81	1	84.91%	833.61%	2.4272%	1.8173%
10	1.87	1	82.86%	916.48%	2.4736%	1.8799%
11	1.925	1	80.80%	997.28%	2.5488%	1.9379%
12	1.98	1	78.70%	1075.97%	2.6770%	1.9965%
13	2.023	1	76.68%	1152.65%	2.6314%	2.0427%
14	2.066	1	74.64%	1227.29%	2.7359%	2.0896%
15	2.11	1	72.57%	1299.86%	2.8428%	2.1372%
16	2.138	1	70.70%	1370.56%	2.6528%	2.1673%
17	2.166	1	68.82%	1439.38%	2.7236%	2.1978%
18	2.194	1	66.95%	1506.33%	2.7960%	2.2289%
19	2.222	1	65.08%	1571.42%	2.8701%	2.2606%
20	2.25	1	63.22%	1634.64%	2.9460%	2.2927%
21	2.256	1	61.73%	1696.37%	2.4149%	2.2971%
22	2.262	1	60.26%	1756.63%	2.4309%	2.3019%
23	2.268	1	58.83%	1815.46%	2.4472%	2.3069%
24	2.274	1	57.41%	1872.87%	2.4637%	2.3122%
25	2.28	1	56.02%	1928.89%	2.4806%	2.3178%
26	2.284	1	54.69%	1983.58%	2.4251%	2.3208%
27	2.288	1	53.39%	2036.98%	2.4366%	2.3240%
28	2.292	1	52.12%	2089.10%	2.4483%	2.3274%
29	2.296	1	50.87%	2139.96%	2.4603%	2.3309%
30	2.3	1	49.64%	2189.60%	2.4724%	2.3346%

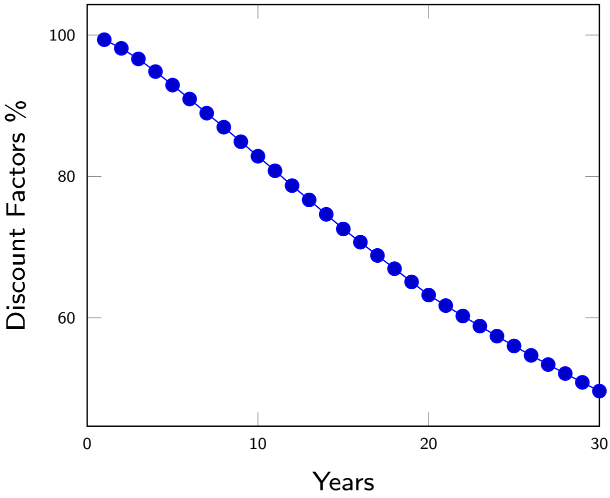

Figure 2: Bootstrapped Discount Curve from Swap Rates

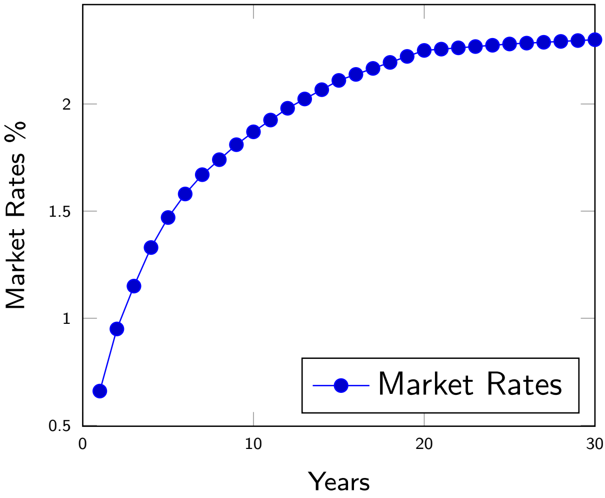

Figure 3: Market Swap Rates

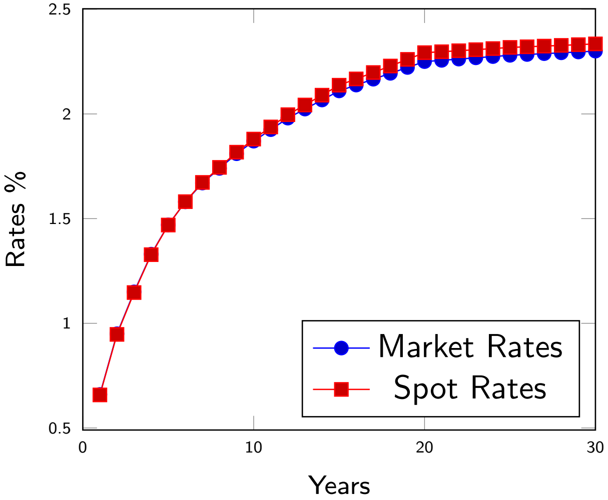

Figure 4: Market Swap Rates

Dealing with Missing Quotes

Linear Interpolation of Swap Rates

- In the interpolation procedure a common assumption consists of assuming that the par swap rate at each intermediate coupon date lies on the straight line joining the adjacent swap rates.
- For example if the 5yr swap rate is 3% and the 10yr swap rate is 4%, the six year swap rate is

$$\frac{10-6}{10-5} \times 3\% + \frac{6-5}{10-5} \times 4\%.$$

- This interpolation method is named, Linear Swap Rates (LSR).
- In the previous example, this assumption has been used to obtain the 11, 13, 16, 17, 18, 19, 21, 22, 23, 24, 26, 27, 28 and 29 swap rates.
- The interpolation method is not neutral on the shape of the forward rates.

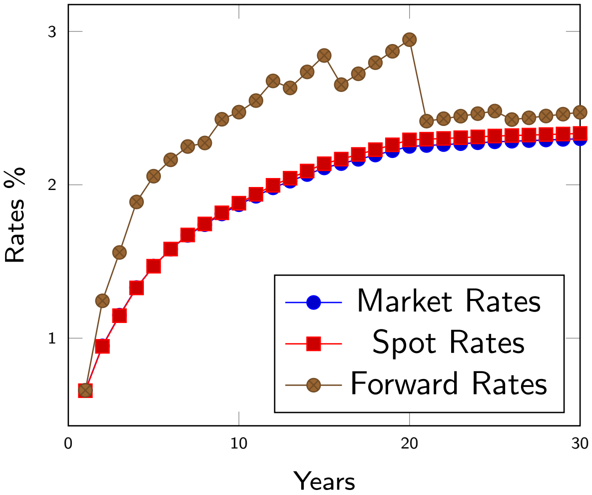

Figure 5: Bootstrapped Term Structures of Spot and Forward Rates (1 year tenor).

Constraining the Forward Curve

- A second method, Constant Forward Rates (CFR), constraints the problem by enforcing that missing swap rates are chosen so that all one year forward rates be constant.
- This method is nowadays fairly standard and it is the simplest market standard methodology. It requires the solution of a simple linear equation.
- Other possibilities consist in constraining Forward Rates to lie on a line or on a parabola.
- An useful discussion can be found at http://www.fincad.com/news/assets/pdfs/dec05/curve_building.pdf

Example (1. Market Quotes)

Accompanying Excel file: FI_Bootstrappings, Sheet: LSV vs CFR

Excel spreadhseet: **LSW vs CFR**

Let us suppose to have the following information

Table 13: Market Rates

Term	Par Swap Rate
1	Not Quoted
2	2.70%
3-4	Not Quoted
5	3.60%
6-7-8-9	Not Quoted
10	4.60%
11-14	Not Quoted
15	4.80%
16-19	Not Quoted
20	4.80%
21-24	Not Quoted
25	4.75%

Example (2. Assume Constant Forward Rate)

Term	0	1	2
F	-	$F(0,0,1)$	$F(0,1,2)$
DF	1	$P(0,1)$	$P(0,2)$
Swap Rate	-	✗	2.70%

- Let us set

$$F = F_{0\times 1} = F_{1\times 2},$$

and we constraint F to give back the quoted swap rate

$$2.70\% = \frac{1 - P(0,2)}{P(0,1) + P(0,2)}.$$

- We have that

$$P(0,1) = \frac{1}{1 + F \times (1-0)} = \frac{1}{1+F},$$

and

$$P(0,2) = \frac{1}{1 + F \times (1-0)} \times \frac{1}{1 + F \times (2-1)} = \frac{1}{(1+F)^2},$$

Example (2.1 Solving a second-order equation)

- Setting $x = \frac{1}{1+F}$, we have

$$2.70\% = \frac{1-x^2}{x+x^2}$$

- We have to solve the equation

$$(1+2.70\%)x^2 + 2.70\% x - 1 = 0,$$

- We find

$$x = 0.9737,$$

and therefore

$$P(0,1) = x = 0.97371, \quad P(0,2) = x^2 = 0.948111.$$

and moreover the forward rate is i.e. $F = 2.70\%$..

- In addition, we re-obtain exactly the 2 year swap rate

$$S(1) = \frac{1}{0.97371} - 1 = 2.70\%, \quad S(2) = \frac{1-0.97371}{0.97371+0.948111} = 2.7\%.$$

Example (3. Solving a second-order equation)

Term	0	1	2	3	4	5
F		2.70%	2.70%	F_3	F_3	F_3
DF	1	0.97371	0.948111	$P(0,3)$	$P(0,4)$	$P(0,5)$
Swap Rate		2.70%	2.70%	✗	✗	3.60%

- Then we have to find F_{2x3}, F_{3x4} and F_{4x5} and we let F_3 to be their common value.
- The 3 year discount factor $P(0,5)$ becomes

$$P(0,3) = P(0,2) \cdot \frac{1}{1 + F_3 \cdot (3-2)} = P(0,2) \cdot x.$$

where $x = \frac{1}{1+F_3 \cdot 1}$.

- The 4 year discount factor $P(0,5)$ becomes

$$P(0,4) = P(0,2) \cdot \frac{1}{1 + F_3 \cdot (3-2)} \cdot \frac{1}{1 + F_3 \cdot (4-3)} = P(0,2) \cdot x^2,$$

Example

- The 5 year discount factor $P(0,5)$ becomes

$$P(0,2)\cdot\frac{1}{1+F_3\cdot(3-2)}\cdot\frac{1}{1+F_3\cdot(4-3)}\cdot\frac{1}{1+F_3\cdot(5-4)} = P(0,2)\cdot x^3$$

- We have to recover the quote of the 5-years swap rate by solving (numerically) the equation

$$3.60\% = \frac{1-P(0,2)\cdot x^3}{P(0,1)+P(0,2)\cdot(1+x+x^2+x^3)}$$

where $P(0,1)=\frac{1}{1+2.7\%}=0.97371$, $P(0,2)=\frac{1}{(1+2.7\%)^2}=0.94811.$

- We obtain the 3th order equation

$$3.60\%\cdot\left(0.97371+0.94811(1+x+x^2+x^3)\right)=1-0.94811\cdot x^3$$

Example

- Solving numerically, we can find $x = 0.95913$ and therefore

$$P(0,3) = 0.94811 \cdot 0.95913 = 0.90937,$$
$$P(0,4) = 0.94811 \cdot (0.95913)^2 = 0.872205,$$
$$P(0,5) = 0.94811 \cdot (0.95913)^3 = 0.83656$$

- Also the corresponding forward rates are:

$$F_{2x3} = F_{3x4} = F_{4x5} = \left(\frac{1}{0.95913} - 1\right) = 4.2606\%$$

- In addition, we obtain the interpolated 3 and 4 years swap rates and we reproduce the 5 years swap rate

$$S_3 = 3.201\%, \quad S_4 = 3.451, \% \quad S_5 = 3.60\%.$$

Example

Term	1	2	3	4	5	6	7	8	9	10
F	2.70%	2.70%	4.261%	4.261%	4.261%	F_5	F_5	F_5	F_5	F_5
DF	97.37%	94.81%	90.94%	87.22%	83.66%	P_6	$\cdots$	$\cdots$	$\cdots$	P_{10}
A	0.9737	1.9218	2.8312	3.7034	4.5400					
S	2.70%	2.70%	3.201%	3.451%	3.6%	✗	✗	✗	✗	4.60%

- We set

$$F_5 = F_{5x6} = F_{6x7} = F_{7x8} = F_{8x9} = F_{9x10}$$

- We have the equation

$$P(0,10) = \frac{1 - S_{10} \sum_{i=1}^{9} P(0,i)}{1 + S_{10}},$$

where $S_{10} = 4.6\%$ and

$$P(0,i) = P(0,5) \cdot \frac{1}{(1+F_5)^{i-5}}$$

- The solution is $F_5 = 5.8843\%$.

Using a similar procedure we can bootstrap the discount curve up to 25 years.

Start	End	CFR	DF	Annnuity	Swap Rate
0	1	**2.7000%**	0.97371	0.97371	2.7000%
1	2	2.7000%	0.948111	1.921821	2.7000%
2	3	**4.2606%**	0.909366	2.831187	3.2013%
3	4	4.2606%	0.872205	3.703391	3.4508%
4	5	4.2606%	0.836562	4.539953	3.6000%
5	6	**5.8843%**	0.790072	5.330025	3.9386%
6	7	5.8843%	0.746165	6.07619	4.1775%
7	8	5.8843%	0.704699	6.780889	4.3549%
8	9	5.8843%	0.665537	7.446425	4.4916%
9	10	5.8843%	0.628551	8.074977	4.6000%
10	11	**5.4000%**	0.596348	8.671325	4.6550%
11	12	5.4000%	0.565795	9.237119	4.7007%
12	13	5.4000%	0.536807	9.773926	4.7391%
13	14	5.4000%	0.509304	10.28323	4.7718%
14	15	5.4000%	0.483211	10.76644	4.8000%
15	16	**4.8000%**	0.461079	11.22752	4.8000%
16	17	4.8000%	0.439961	11.66748	4.8000%
17	18	4.8000%	0.41981	12.08729	4.8000%
18	19	4.8000%	0.400582	12.48787	4.8000%
19	20	4.8000%	0.382235	12.87011	4.8000%
20	21	**4.3679%**	0.366238	13.23635	4.7880%
21	22	4.3679%	0.35091	13.58726	4.7772%
22	23	4.3679%	0.336224	13.92348	4.7673%
23	24	4.3679%	0.322153	14.24563	4.7583%
24	25	4.3679%	0.308671	14.5543	4.7500%

Table 14: Bootstrapped Discount Curve via Constant Forward Rate Methods

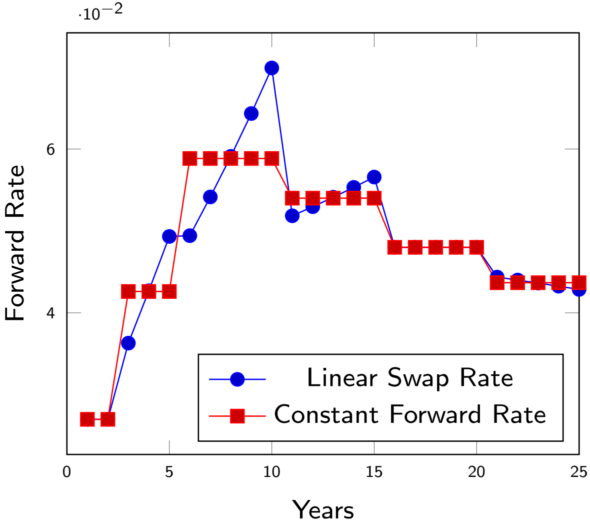

Figure 6: Constant Forward Rates versus Forward Curve built via linear interpolation of swap rates.

Question

Let us suppose that:

- the 5 yrs discount factor is 0.95;
- the annuity up to 5 years is 4.85;
- the 5 yrs swap rate is 1.0309%;
- the 7 year swap rate is 1.08%;

Determine the 7 year discount factor using the linear swap rate and constant forward rate methods.

Answer: linear swap rate method

Using the linear swap rate method, we have

- The 6 year swap rate is 0.01055% (=(1.0309+1.08)/2).
- The 6 year discount factor is

$$P(0,6) = \frac{1 - 1.055\% \cdot 4.85}{1 + 1.055\%} = 0.9389.$$

- The 6 year annuity is

$$A_6 = 4.85 + 0.9389 = 5.7889.$$

- The 7 year discount factor is

$$P(0,7) = \frac{1 - 1.08\% \cdot 5.7889}{1 + 1.08\%} = 0.927463.$$

- The last two forward rartes, i.e. 5x6 and 6x7, are equal to 1.1822% and 1.2331%.

Answer: constant forward rate method I

Using the constatt forward rate method, we have

- to determine the F_{5x6} and F_{6x7} forward rates and their common value F_6 such that we reobtain quoted swap rates.
- The 6 and 7 year discount factor are

$$P(0,6) = P(0,5) \cdot \frac{1}{1+F_6}, \text{ and } P(0,7) = P(0,5) \cdot \frac{1}{(1+F_6)^2}.$$

- The equilibrium condition is

$$1.08\% = \frac{1 - P(0,7)}{A_5 + P(0,6) + P(0,7)},$$

- Set $x = 1/(1+F)$, so that we have

$$P(0,6) = P(0,5) \cdot x, \text{ and } P(0,7) = P(0,5) \cdot x^2.$$

Answer: constant forward rate method II

- Imposing that the discount factors reprice exactly the swap rate, we have

$P(0,5)x^2$

$$S_7 = 1.08\% = \frac{1 - P(0,5)\cdot x}{A_5 + P(0,5)\cdot x + P(0,5)\cdot x^2},$$

or, equivalently

$$1.08\% \cdot \left(A_5 + P(0,5)\cdot x + P(0,5)\cdot x^2\right) = 1 - P(0,5)\cdot x$$

i.e.

$$1.08\% = \frac{1 - 0.95\cdot x}{4.85 + 0.95\cdot(x + x^2)}.$$

- We can solve numerically the above equation and obtain

$$x = 0.98807$$

and we can then recover the 6 and 7 year discount factors

$$P(0,6) = 0.93867, \text{ and } \quad P(0,7) = 0.927466.$$

Answer: constant forward rate method III

- In addition, the forward rates 5x6 and 6x7 are equal to

$$F_6 = 1.2075\%.$$

Conclusions

- The discount rates provide the tools by which we can now analyse deals and dealing positions.
- The value of any deal in a portfolio will be the sum of the present values of each individual cash flow of which that deal comprises.
- However, the discount curve has been computed only at the grid points. In many cases, cash flows will not fall on the node points.
- Some methodology for connecting the nodes to calculate discount factors for all time period is required.
- Linear interpolation is reputed too inaccurate and significant resources need to be allocated to provide an interpolation methodology.

Term Structure Fitting

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MSc Quantitative Finance

SMM269 Fixed Income

Academic Year 2019-20

References

Useful Readings

- Bruce Tuckman, Angel Serrat. Fixed Income Securities: Tools for Today's Markets, 3rd Edition, **Chapter 21. Curve Construction.** 16th May 2013.
- Frank Fabozzi, Fixed Income Analysis, pages 193-196.

Excel Files

- FI_Bootstrapping.xlsm
- FI_FittingNelsonSiegel.xlsm

Outline

1. **How to interpolate the spot term structure**
   - Linear Interpolation
   - Geometric Interpolation
   - Parametric Interpolation
   - Non-Parametric Interpolation of the discount curve

Application to trades and positions

- The discount rates provide the tools by which we can now analyse deals and dealing positions.
- The value of any deal in a portfolio will be the sum of the present values of each individual cash flow of which that deal comprises.
- However, the discount curve has been computed only at the grid points. In many cases, cash flows will not fall on the node points.
- Therefore, once we have constructed the discount curve at grid points we would like to extend it to some other points. For example the dates where a bond pays its coupons.
- Some methodology for connecting the nodes to calculate discount factors for all time period is required.
- Linear interpolation is reputed too inaccurate and significant resources need to be allocated to provide an interpolation methodology.

Linear Interpolation

- Let us suppose that after bootstrapping, we have computed $P(t, T_1)$ and $P(t, T_2)$, and we are interested in $P(t, T)$, where $T_1 < T < T_2$.
- **Linear interpolation**. Given the discount factors, we choose a compounding convention, e.g. continuous, and we compute the corresponding spot rates and then we linear interpolate the spot rate for maturity T and then we get the interpolated discount factor

1 We linearly interpolate the spot rate

$$R(t, T) = \frac{T_2 - T}{T_2 - T_1} R(T, T_1) + \frac{T - T_1}{T_2 - T_1} R(T, T_2)$$

where $R(t, T) = -ln(P(, T))/(T - t)$.

2. We get the interpolated discount factor

$$P(t, T) = e^{-(T-t)R(t,T)}$$

- In general, this procedure gives a very irregular forward curve.

Example

- We are interested in the 70 days discount factor given the information in the following table

Table 1: Market Information

Days	TTM	DF	Spot Rate (c.c.)
60	0.164384	0.999	0.6086%
90	0.246575	0.998	0.8119%

- We have

$$R(t, t + 70days) = \frac{90-70}{90-60} \cdot 0.6086\% + \frac{70-60}{90-70} \cdot 0.8119\% = 0.6764\%$$

and then the interpolated discount factor is

$$P(t, t + 70days) = e^{-\frac{70}{365} \cdot 0.6764\%} = 0.99870.$$

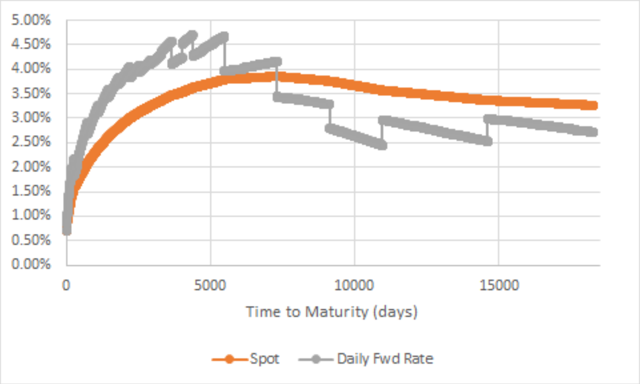


Figure 1: Linearly Interpolated Spot Curve. The Example is detailed in the xls file **FI_bootstrapping.xlsm**, Sheet: **Interpolation Linear**.

Constant Forward Rate Method

- Some improvements, it is obtained by constraining the forward curve (daily tenor) to be piecewise constant.
- Given the two discount factors, let n to be the number of days between T_1 and T_2.
- We impose that the daily forward discount factor to be constant and equal to x, so that

$$P(t, T_2) = P(t, T_1) \prod_{i=1}^{n} P\left(t, T_1 + \frac{i-1}{365}, T_1 + \frac{i}{365}\right) = P(t, T_1) \cdot x^n$$

i.e. $x = \left(\frac{P(t,T_1)}{P(t,T_2)}\right)^{\frac{1}{n}}$ and then we compute

$$P(t, T) = P(t, T_1) \cdot x^m$$

where m is the number of days between T_1 and T.

Example

- We are interested in the 70 days discount factor given the following information

Days	TTM	DF	Spot Rate (c.c.)
60	0.164384	0.999	0.6086%
90	0.246575	0.998	0.8119%

- The two discount factors are 30 days apart. We compute the daily forward discount factor

$$P(t, t+(i-1)days, t+idays) = \left(\frac{0.998}{0.999}\right)^{\frac{1}{30}} = 0.999966617$$

- The interpolated discount factor is 20 days apart from the first discount factor. So

$$P(t, t+70days) = 0.999 \cdot (0.999966617)^{20} = 0.998333.$$

- The interpolated spot rate is
$R(t, t+70days) = -\frac{\ln(P(t,t+70days))}{70/365} = -\frac{ln(0.998333)}{70/365} = 0.8698\%.$

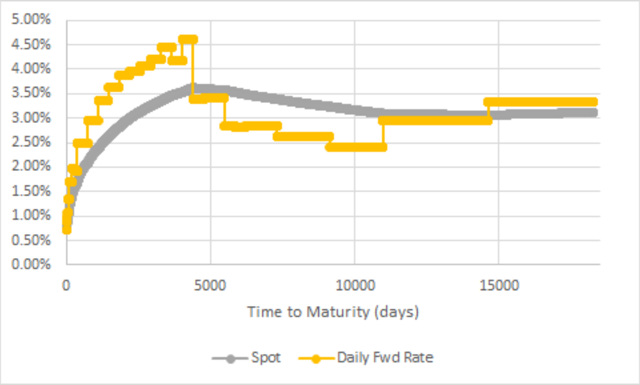


Figure 2: Interpolated Spot Curve using the Piecewise Constant Forward Method. The Example is detailed in the xls file **FI_bootstrapping.xlsm**, Sheet: **Interpolation Cst Fwd**.

Interpolating the discount curve

- Two more general approaches have been proposed: the parametric and the non-parametric methods.
- The first method models the forward curve by a parametric function. The parameters are fitted using a minimization routine.
- With the non-parametric method, we use a piecewise polynomial function (e.g. a piecewise cubic polynomial), and we join the so-called knot points, where the function and its first derivative are continuous. The polynomial is eventually constrained to guarantee some smoothness in the forward curve.
- Ioannides (2003) provides a detailed comparison among the different estimation techniques.

Interpolating Discount Curve via Parametric Approach

- A popular approach is to postulate a parametric functional form for the bootstrapped discount curve $P(t, T; \boldsymbol{\theta})$ as function of time to maturity and some parameter set $\boldsymbol{\theta}$ and we use the market observed discount curve to **estimate** (calibrate) the parameters of this functional form.
- One popular parametric model is the **Nelson&Siegel model** (1987), for which the discount curve is assumed to be as follows

$$P_{NS}\left(t, t+\tau; \boldsymbol{\theta}\right) = \exp\left(-\tau \times R_{NS}\left(t, t+\tau; \boldsymbol{\theta}\right)\right).$$

where the continuously compounded spot rate is defined according to

$$R_{NS}\left(t, t+\tau; \boldsymbol{\theta}\right) = \beta_0 + \left(\beta_1 + \frac{\beta_2}{k}\right)\frac{1-\exp\left(-\tau k\right)}{\tau k} - \frac{\beta_2}{k}\exp\left(-\tau k\right),$$

$$\to \frac{1-(1-\tau k)}{\tau k} \to 1$$

where $\boldsymbol{\theta}$ is the vector of unknown parameters.

$$\boldsymbol{\theta} = \{\beta_0, \beta_1, \beta_2, k\}.$$

Interpreting the Nelson-Siegel parameters

- In the Nelson-Siegel model β_0, β_1, β_2 and κ are the parameters to be estimated.
- In particular, the short and long end of the curve are related to the NS parameters:
  - β_0 specifies the long rate to which the spot rate tends asymptotically
  $$\lim_{\tau\to\infty} R_{NS}(t, t+\tau; \boldsymbol{\theta}) = \beta_0.$$
  - β_1 is the weight attached to the short term component (spread short/long-term)
  $$\lim_{\tau\to 0} R_{NS}(t, t+\tau; \boldsymbol{\theta}) = \beta_0 + \beta_1.$$
  - β_2 is the weight attached to the medium term component.
  - k measures the point of the beginning of decay.
- You can figure out how the different parameters affect the shape of the term structure playing with the Excel file **FI_FittingNelsonSiegel.xls** sheet **NelsonSiegel**.

Calibrating the Nelson-Siegel curve to swap market

- Given the constructed discount curve at the n market grid points, compute the continuously computed spot rate $R_{mkt}(t, t+\tau)$ and choose the parameters that minimize some distance function between $R_{mkt}(t, t+\tau)$ and $R_{NS}(t, t+\tau; \boldsymbol{\theta})$:

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} w_i \left(R_{NS}(t, t+\tau_i; \boldsymbol{\theta}) - R_{mkt}(t, t+\tau_i) \right)^2 .$$

- The fitting could also be defined in terms of price rather than yield errors.

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} w_i \left(P_{NS}(t, t+\tau_i; \boldsymbol{\theta}) - P_{mkt}(t, t+\tau_i) \right)^2 .$$

- This is the preferred approach when we deal with bond markets data.
- The weights w_i can be chosen equal or to give more importance to specific maturities (eg. the most liquid ones).

Case Study

Excel File: **FI**$_F$*ittingNelsonSiegel*
Excel sheet: FittingBootstrappedCurve
Aim:

1. **Bootstrap market discount curve;**
2. **Interpolate the discount curve;**
3. **Price a bond expiring in 5 years with quarterly payments. Coupon rate is 1.50%.**

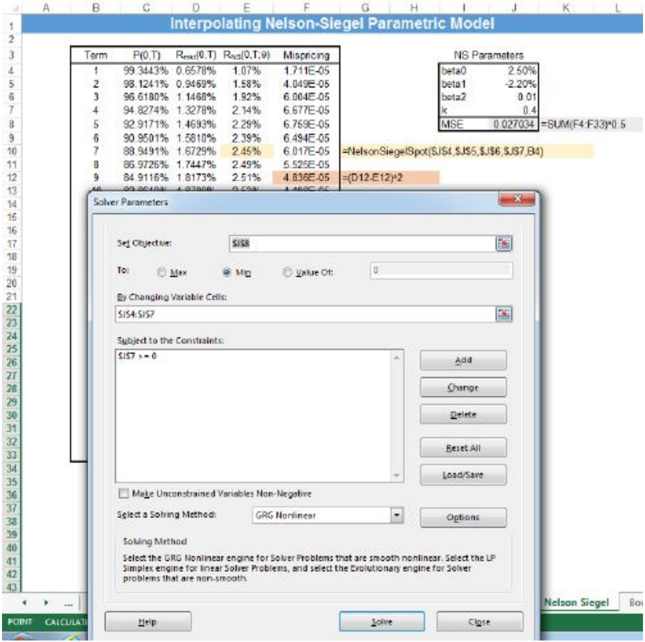


Figure 3: Interpolating the Nelson-Siegel curve using the Excel Solver

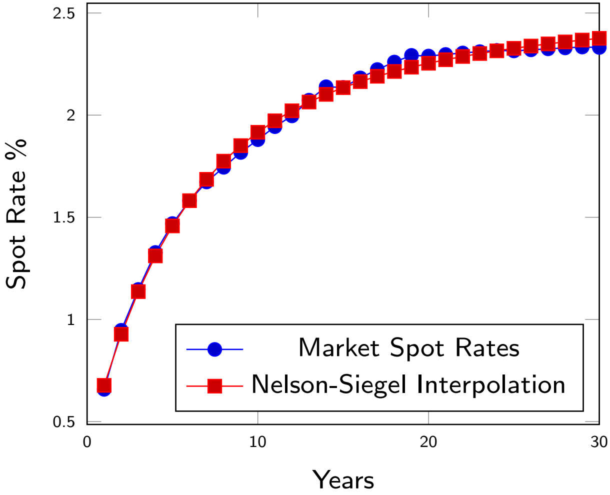


Figure 4: Market and interpolated spot rates.

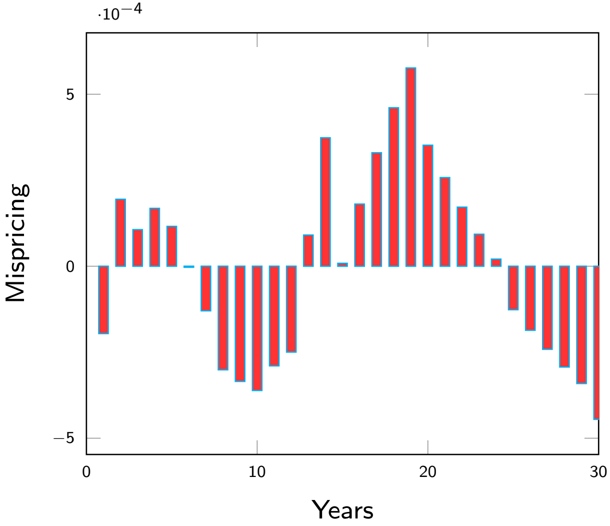


Figure 5: Difference between Market and Fitted spot rates.

Bond Pricing

T	Cpn Tenor	Cpn Rate	Cash Flows	Spot Rate NS	DF NS	PV(CF)	
0.25	0.25	1.5	0.375	0.46%	99.89%	0.374571	
0.5	0.25	1.5	0.375	0.53%	99.73%	0.373999	=NelsonSiegelSpot($J$4;$J$5;$J$6;$J$7;B41)
0.75	0.25	1.5	0.375	0.61%	99.55%	0.373295	
1	0.25	1.5	0.375	0.68%	99.32%	0.372468	
1.25	0.25	1.5	0.375	0.74%	99.07%	0.371528	=EXP(-F44*B44)
1.5	0.25	1.5	0.375	0.81%	98.80%	0.370482	
1.75	0.25	1.5	0.375	0.87%	98.49%	0.36934	
2	0.25	1.5	0.375	0.93%	98.16%	0.368108	=G47*E47
2.25	0.25	1.5	0.375	0.98%	97.81%	0.366796	
2.5	0.25	1.5	0.375	1.04%	97.44%	0.365408	
2.75	0.25	1.5	0.375	1.09%	97.05%	0.363952	
3	0.25	1.5	0.375	1.14%	96.65%	0.362433	
3.25	0.25	1.5	0.375	1.18%	96.23%	0.360858	
3.5	0.25	1.5	0.375	1.23%	95.79%	0.359231	
3.75	0.25	1.5	0.375	1.27%	95.35%	0.357558	
4	0.25	1.5	0.375	1.31%	94.89%	0.355842	
4.25	0.25	1.5	0.375	1.35%	94.42%	0.354089	
4.5	0.25	1.5	0.375	1.39%	93.95%	0.352302	
4.75	0.25	1.5	0.375	1.42%	93.46%	0.350485	
5	0.25	1.5	100.375	1.46%	92.97%	93.31955	
					PV	**100.2423**	

Figure 6: Pricing the coupon bond using the interpolated Nelson-Siegel curve

Calibrating the Nelson-Siegel curve to Bond Prices

1. For a given set of parameters, we can compute the bond price implied by the NS model

$$B_{NS}(c,t,T;\boldsymbol{\theta}) = \sum_{j=1}^{n} \frac{c}{m} \times e^{-(T_j-t)\times R_{NS}(t,T_j,\boldsymbol{\theta})} + e^{-(T_n-t)\times R_{NS}(t,T_n,\boldsymbol{\theta})}$$

2. For the same bond, we have the invoice price quoted in the market $B_{mkt}(c,t;T)$.
3. For each given set of parameters $\boldsymbol{\theta}$ and considering bonds with different maturities, we can compute the mean-square error

$$MSE\,(\boldsymbol{\theta}) = \sum_{i=1}^{N} \left(B_{mkt}(c,t;T_i) - B_{NS}(c,t,T_i;\boldsymbol{\theta})\right)^2.$$

4. The procedure is then to find parameters $\boldsymbol{\theta}$ such that the MSE is minimized

$$\hat{\boldsymbol{\theta}} : \text{argmin} MSE\,(\boldsymbol{\theta})$$

5. This is the approach followed by the European Central Bank.[1]

[1] BCE, Technical Notes, http://www.ecb.europa.eu/stats/money/yc/html/technical_notes.pdf.

Case Study

Excel File: **FI**$_F$*ittingNelsonSiegel*
Excel sheet: FittingBondPrices
Aim:

1. **Build the Bond Cash Flow Matrix;**
2. **Price the Bonds according to NS;**
3. **Minimize the Sum of Squared Errors.**

Example (1. The Cash Flow Matrix)

Table 2: Bond Cash Flows

		Cash Flows									
Bond	Gross Price	1	2	3	4	5	6	7	8	9	10
1	95.6	1	101	0	0	0	0	0	0	0	0
2	93.4	1.2	1.2	101.2	0	0	0	0	0	0	0
3	95.1	2.5	2.5	2.5	102.5	0	0	0	0	0	0
4	92.7	2.8	2.8	2.8	2.8	2.8	102.8	0	0	0	0
5	92.9	3	3	3	3	3	3	103	0	0	0
6	92.5	3.1	3.1	3.1	3.1	3.1	3.1	3.1	103.1	0	0
7	87.1	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	102.5	0
8	101.8	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	104.5

Example (2. Assign Initial Values to the NS parameters)

Table 3: Starting values of the NS model and corresponding NS discount curve

NS Parameters	
β_0	0.04
β_1	-0.02
β_2	0.015
κ	0.45

TTM	NS Discount
1	0.97384
2	0.93813
3	0.89861
4	0.85852
5	0.81957
6	0.78255
7	0.74771
8	0.71505
9	0.68444
10	0.65570

Example (3. Reprice Bonds using the NS Discount Curve)

We can compute NS bond prices via the matrix product

$$\boldsymbol{CF} \cdot \boldsymbol{DF_{NS}}$$

where: $\boldsymbol{CF}$ is the NxM cash flow matrix (in our example N=nr. of bonds=5; M=number of maturities=10) and $\boldsymbol{DF_{NS}}$ is the $Mx1$ vector containing the NS theoretical bond prices.

Bond	Bond Price	NS Price	Pricing Error	Weigth	$Error^2$
1	95.6	95.0977	0.5023	1.0000	0.2523
2	93.4	92.3974	1.0026	1.0000	1.0053
3	95.1	94.0933	1.0067	1.0000	1.0135
4	92.7	92.1984	0.5016	1.0000	0.2516
5	92.9	92.1200	0.7800	1.0000	0.6084
6	92.5	91.784933	0.7151	1.0000	0.5113
7	87.1	86.531125	0.5689	1.0000	0.3236
8	101.8	101.41661	0.3834	1.0000	0.1470

Example (4. Execute the Minimization of the SSE (e.g using Excel Solver))

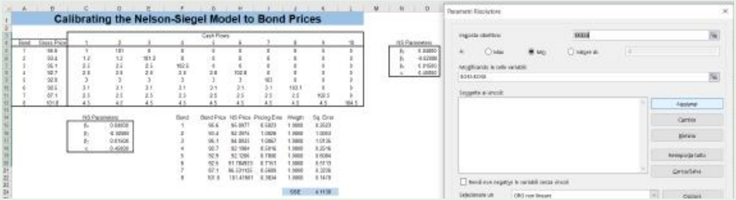


Figure 7: Setting up the minimization using the Excel Solver

Example (5. Check the results)

Table 5: Results of the calibration of the Nelson Siegel Model

NS Parameters		Bond	Bond Price	NS Price	Pricing Error	Weight	$Error^2$
β_0	0.03879	1	95.6	95.7206	-0.1206	1.0000	0.0145
β_1	-0.02002	2	93.4	93.2307	0.1693	1.0000	0.0287
β_2	0.01199	3	95.1	95.0228	0.0772	1.0000	0.0060
k	0.33035	4	92.7	93.0118	-0.3118	1.0000	0.0972
		5	92.9	92.8248	0.0752	1.0000	0.0057
		6	92.5	92.377719	0.1223	1.0000	0.0150
		7	87.1	86.988895	0.1111	1.0000	0.0123
		8	101.8	101.90422	-0.1042	1.0000	0.0109
						SSE	0.1902

Example (5. Check the results: Pricing Errors)

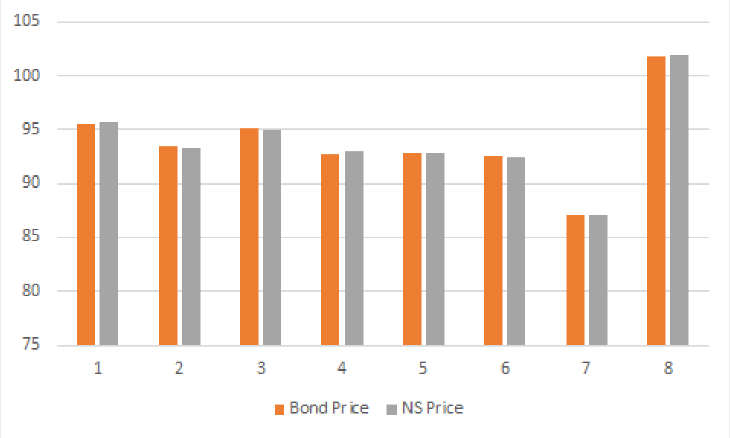


Figure 8: Market and Model Bond Prices

Example (5. Check the results: Bond Prices)

Figure 9: Errors in repricing the bonds

Example (5. Check the results: The Nelson Siegel fitted curve)

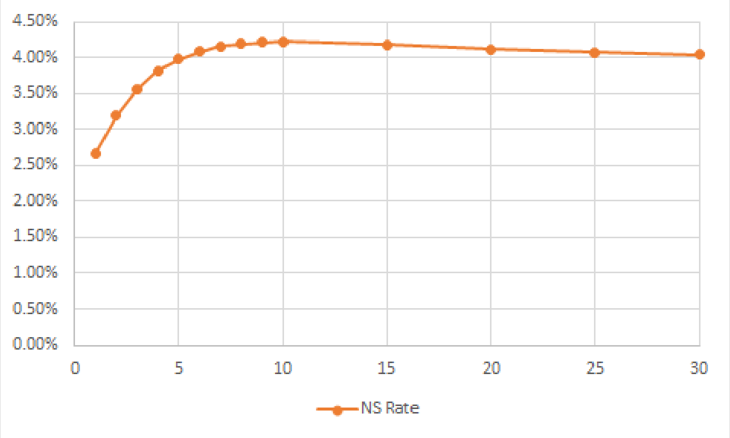


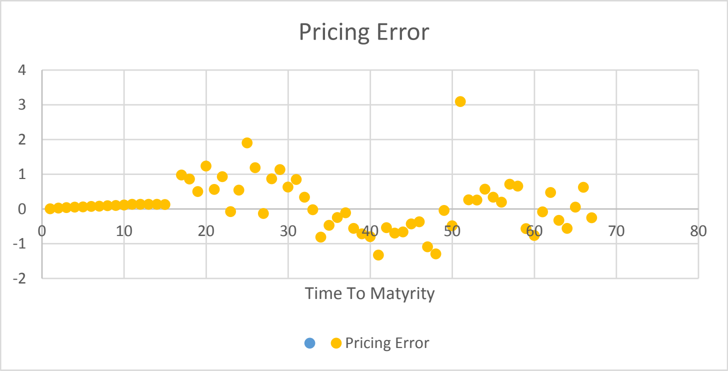
Figure 10: Fitted Nelson-Siegel curve

Case Study: Calibrating the Nelson-Siegel curve to the BTP market

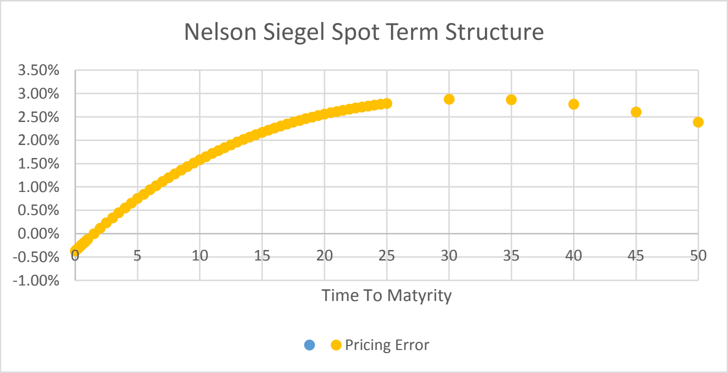
Excel File: **FI**$_F$*ittingNelsonSiegel*
Excel sheet: CaseStudy Bond Fitting
Aim: Fitting the Nelson Siegel model to market bond prices

1. **Build the Bond Cash Flow Matrix;**
2. **Price the Bonds according to NS;**
3. **Minimize the Sum of Squared Errors.**

Curve Fitting



Curve Fitting



Parsimonious Models – Pros and Cons

- We can capture as much as possible of the term structure with a few parameters as possible.
- There is a balance between goodness of fit on the one hand and parsimony on the other: they lack of flexibility, i.e. cannot account for all possible shapes of the TS we see in practice.
- Alternative approach like spline models are more flexible, better for pricing but much less parsimonious.

Cubic Spline

- The cubic spline parameterization was first used by McCulloch (1971) to estimate the nominal term structure. He later added taxes in McCulloch (1975).
- The methodology is described for instance in Fabozzi, Interest rate, term structure, and valuation modeling, from page 158 to page 183.
- The interpolating function is a piecewise cubic function.
- The cubic function is chosen to guarantee continuity up to the second derivative of the spot curve at joining points.
- A Youtube presentation can be found at `http://wn.com/Fit_Treasury_yield_curve_with_cubic_polynomial`.

Main References I

1. Adams K. J. and Donald R. van Deventer, 1994, Fitting Yield Curves and Forward Rate Curves with Maximum Smoothness, The Journal of Fixed Income, June 1994, 52-62.
2. Bank for International Settlements (2005). Zero-Coupon Yield Curves: Technical Documentation. BIS Papers 25, Monetary and Economic Department. `http://www.bis.org/publ/bppdf/bispap25.htm`.
3. Fabozzi F. (ed.), Interest Rate, Term Structure, and Valuation Modeling (Hardcover), Wiley; 1st edition (July 15, 2002).
4. Hagan PS, West G (2006). Interpolation Methods for Curve Construction. Applied Mathematical Finance, 13(2), 89(129).
5. Ioannides M., A comparison of yield curve estimation techniques using UK data, Journal of Banking and Finance, 27, 2003, 1-26.
6. James J. and N. Webber, Interest Rate Modelling, Wiley series in Financial Engineering, 2000.
7. Martellini L., Priaulet P. and Priaulet S., Fixed Income Securities, Wiley Finance, 2003. `Chapter 4`.

Main References II

8. McCulloch J. Huston, 1977, Measuring the Term Structure of Interest Rates, Journal of Business, 44(1), 19-31.
9. McCulloch J. Huston , 1975, The Tax Adjusted Yield Curve, Journal of Finance 30, 811-29.
10. Nelson C. and A. F. Siegel, 1987, Parsimonious Modeling of Yield Curve, Journal of Business, 60(4), 473-489.
11. Steeley James M. , Estimating the Gilt-Edged Term Structure: Basis Splines and Confidence Intervals, Journal of Business Finance and Accounting (June 1991), 512-529.

Interest Rate Modelling: Main Issues

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MSc Financial Mathematics
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MSc Quantitative Finance

SMM269 Fixed Income

Academic Year 2019-20

General Requirements for a model

General Requirements for a model

- Accurate Valuation of Simple Market Instruments
- Easy calibration to the market
- Robustness, i.e. the model should perform well in all markets
- Extensibility to new instruments
- Stability of floating parameters

The Uses of Interest rate models

- Pricing and hedging
- Risk Management
- Explaining interest rate movements.
- So we will need techniques for
  - Describing interest rate movements
  - Obtaining prices from models
  - Estimating parameters

Basic Steps

- Decide upon the state variables (which and how many):
  - bond price, short rate, spot rate, fwd rate
  - "pure" state variables
  - economic variables from equilibrium considerations.
- Decide upon the dynamics of the state variable:
  - e.g. discrete/continuous.
  - diffusion/jump.
- Decide upon an appropriate valuation method
  - analytical, numerical, simulation.
- Decide upon the parameter estimation method
  - in which way and to what calibrate the model to market prices.

Criteria for Model Selection

- Fitting market data
  - the current term structure
  - current caplet/bond option prices
  - current volatility structure (e.g. the t.s. of bond option implied volatilities).
- Good dynamics, such as:
  - mean reversion and volatility of the short rate
  - non-negativity of interest rates
  - number and shapes of the future term structures
- Tractability: is there analytical solution ?
  - analytical solution for basic instruments
  - simple numerical solution for exotic derivatives.

Basic models

- Affine yield models (Duffie and Kan, 1994)
  - generalize to n variables the short rate models of Vasicek and CIR.
- Consistent models (Heath, Jarrow and Morton, 1992 and Black, Derman and Toy, 1990):
  - allow for perfect fit of the term structure of interest rate and of the term structure of volatility.
- Market models (Brace et al. 1997 and Jamshidian 1996):
  - the state variables are market quoted rates and justify the use of the Black model in the common practice.

Term Structure vs. asset price modelling

- We need to model a structure of prices.
- We need to model in a consistent way zcb with different expiry (no arbitrage across maturities).
- We need to satisfy the maturity constraint $P(T, T) = 1$.
- The volatility of bond prices behaves differently from asset price volatility:

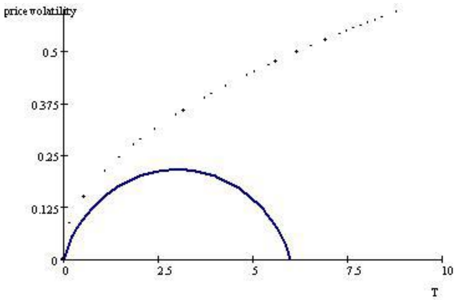
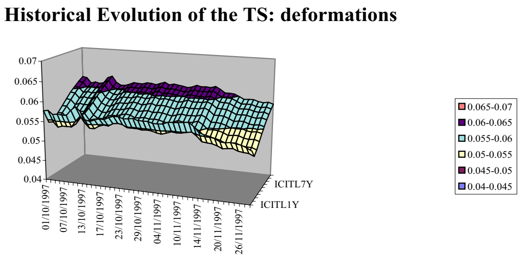
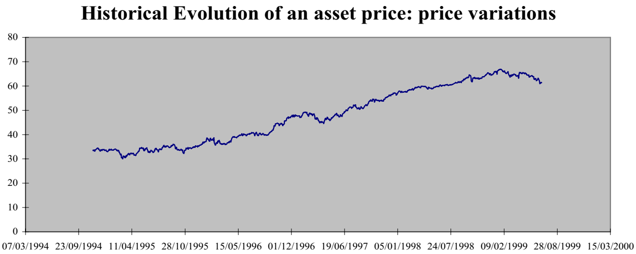


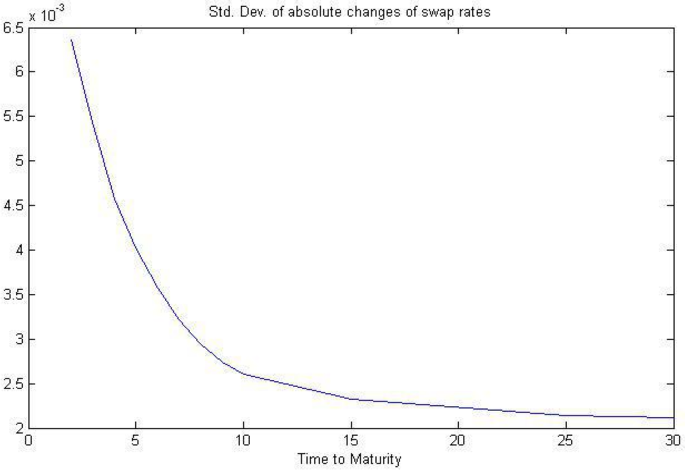
Figure: zcb volatility (blue line) vs. stock volatility (dotted line)

Term Structure Dynamics vs. Stock Price Dynamics





The term structure of interest rate volatility



Correlation of changes in rates

Table: Correlation of changes in the US yields

	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
1 Yr	100%	75%	74%	71%	67%	62%	56%	52%
2 Yr	75%	100%	94%	91%	87%	82%	73%	68%
3 Yr	74%	94%	100%	96%	93%	90%	81%	76%
5 Yr	71%	91%	96%	100%	98%	95%	88%	83%
7 Yr	67%	87%	93%	98%	100%	98%	93%	89%
10 Yr	62%	82%	90%	95%	98%	100%	96%	93%
20 Yr	56%	73%	81%	88%	93%	96%	100%	97%
30 Yr	52%	68%	76%	83%	89%	93%	97%	100%

Which quantity to model?

- The instantaneous short rate;
- The term structure of instantaneous forward rates;
- The term structure of simple forward rates;
- The term structure of discount factors.

Choosing the state variables

- Let us recall the basic relationship for continuously compounded interest rates:

	$P(t,T)$	$R(t,T)$	$f(t,T)$
$P(t,T)$	-	$e^{-R(t,T)(T-t)}$	$e^{-\int_t^T f(t,s)ds}$
$R(t,T)$	$-\frac{\ln P(t,T)}{T-t}$	-	$\frac{\int_t^T f(t,s)ds}{T-t}$
$f(t,T)$	$-\frac{\partial \ln P(t,T)}{\partial T}$	$R(t,T)+(T-t)\frac{\partial R(t,T)}{\partial T}$	-
$r(t)$	$-\left.\frac{\partial \ln P(t,T)}{\partial T}\right\|_{T=t}$	$R(t,t)$	$f(t,t)$

- Which variable is convenient to model?

Introduction to Short Rate Modelling

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MSc Quantitative Finance

SMM269 Fixed Income

Academic Year 2019-20

References

Useful Readings

- Brigo - Mercurio, chapter
- Veronesi book, chapters 14-15-16.
- Vasicek Tutorial

Excel Files

- FI_ShortRateModels.xls
- FI_ImplementingVasicek.xls

Outline I

1. **Main quantities**
   - Instantaneous rate
   - Money Market Account

2. **Pricing with stochastic interest rates**

3. **Money Market Account and zcb pricing**

4. **Exogenous short rate models**

5. **Appendix**

Interest rate modelling

- In order to price non linear derivatives, we need to introduce a stochastic model for interest rates.
- Which variable should we model?
  - Short rate r_t;
  - Instantaneous forward rates $f(t, T)$;
  - Libor rates $L(t, T)$;
  - Forward Libor rates $F(t, T_{i-1}, T_i)$;
  - Forward swap rates $S(t, T_0, T_n)$;
  - Zero coupon bond prices $P(t, T)$;
  - Other key variables???

Overview of the classical approach [1]

- 1977: Short rates ⇒ Endogenous short rate models
- 1990: Short rates ⇒ Exogenous short rate models
- 1990: Instantenous forward rates ⇒ HJM models
- 1997: Forward market rates (forward Libor rates, forward swap rates) ⇒ Market models
- 2002: Volatility smile extensions of forward market-rates models:
  - Local volatility models (e.g. CEV model)
  - Stochastic volatility models (e.g. SABR model)

[1] Brigo, Columbia University Seminar, Paradigm shifts in recent years, 2007

Instantaneous rates

- Instantaneous spot and forward rates do not exist as traded quantities in the market.
- They are convenient theoretical concepts used for modelling purposes: the math is simpler.
- The first term structure models were short rate models, i.e. models assigning the dynamics of the instantaneous spot rate.
- The second generation term structure models, the so-called HJM models, were models assigning the dynamics of the instantaneous forward rate.

Instantaneous spot rate

- Consider the continuously compounded spot rate:

$$R(t, T) = -\frac{\ln P(t, T)}{T - t}$$

- By letting $\Delta \to 0$, we define the instantaneous spot rate (or short rate):

$$\begin{aligned} r(t) &= \lim_{\Delta \to 0} r(t, t + \Delta) \\ &= -\lim_{\Delta \to 0} \frac{\ln P(t, t + \Delta) - \ln P(t, t)}{\Delta} \\ &= -\left.\frac{\partial \ln P(t, T)}{\partial T}\right|_{T=t}. \end{aligned}$$

- Note that the (instantaneous) `short rate` $r(t)$ does not depend on the maturity T any longer.
- $r(t)$ represents just a point on the term structure of spot rates: it is the intercept on the vertical axis.

The money market account

- Let us suppose to invest a unitary amount in a bank account with an instantaneous accruing rate $r(t)$.
- The amount $B(t)$ available in the bank account at time t growths according to the equation

$$dB(t) \quad = \quad r(t)\,B(t)\,dt.$$

- Solving the ordinary differential equation, we get an expression for the money market account at time T given an initial investment at time t.

The money market account

$$B(T) = B(t) \times e^{\int_t^T r(s)ds}.$$

Deterministic vs. Stochastic

Deterministic world

$$
\begin{aligned}
P(t,T) &= e^{-\int_t^T r(s)ds}, \\
R(t,T) &= \frac{\int_t^T r(s)\,ds}{T-t}, \\
f(t,T) &= r(T).
\end{aligned}
$$

Stochastic world

$$
\begin{aligned}
P(t,T) &= \tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)ds}\right], \\
R(t,T) &= -\frac{1}{T-t}\ln\tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)ds}\right], \\
f(t,T) &= -\frac{\partial\ln\tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)ds}\right]}{\partial T}.
\end{aligned}
$$

Pricing with stochastic interest rates

No-arbitrage pricing

Bond prices are arbitrage free if and only if there exists a measure (the risk-neutral one), under which, for each T, the discounted price process $P(t,T)/B(t)$ is a martingale: Relative Price

$$\frac{P(t,T)}{B(t)} = \tilde{\mathbb{E}}_t\left(\frac{P(T,T)}{B(T)}\right).$$

- Using the fact that $P(T,T) = 1$, we have:

Pricing formula for zcb

$$P(t,T) = \tilde{\mathbb{E}}_t\left(\frac{B(t)}{B(T)}\right) = \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right).$$

The short rate approach

- How to use the relationship

$$P(t,T) = \tilde{\mathbb{E}}_t\left(\frac{B(t)}{B(T)}\right) = \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right)?$$

1. Assign the dynamics of the short rate $r(s)$, $t < s < T$, under the risk-neutral measure.
2. Compute the distribution of $\int_t^T r(s)\,ds$.
3. Compute the expectation $\tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right)$.

- This is indeed the approach followed in first term structure models, Merton, Vasicek and CIR, that make some assumption on the stochastic behavior of the short rate and then find $P(t,T)$.
- A more modern no-arbitrage approach takes as given the discount curve and the volatility surface (and equal to the market quotation) and looks for a dynamics of market observable quantities that is consistent with such quotations.

Short Rate Models (or equilibrium models)

$dr = \mu\, dt + \sigma\, dW_t$ $\qquad r_t = r_s + \mu(s-t) + \sigma \int_s^T dW_t$

- **Merton (1974)** proposes a model for the instantaneous rate $r(t)$ based on the arithmetic Brownian motion. only parallel shift
- **Vasicek (1977)** constructs a model for the instantaneous rate $r(t)$ allowing for mean-reversion.
- Successive models, e.g. **Cox, Ingersoll and Ross (1985)**, concentrate on this variable looking for more realistic models (e.g. non-negative interest rates, multifactor models, etc.).
- These models, named `equilibrium models`, cannot replicate the observed zcb prices. This is called the `consistency problem`.
- They can be used to find out mispriced zcb prices, but are useless in pricing interest rate derivatives.

Framework of short rate models

- We assign an Ito diffusion to the short rate, under the risk-neutral measure:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t),$$

  where
  - $\mu(t, r(t))$ is the drift coefficient,
  - $\sigma(t, r(t))$ is the diffusion coefficient,
  - $dW(t)$ is the increment of the Wiener process, i.e.

$$dW(t) \sim \mathcal{N}(0, dt).$$

- The zcb dynamics under the risk-neutral measure is:

$$dP(t, T) = r(t)P(t, T)dt + \nu(t, r(t))P(t, T)dW(t),$$

- All contingent claims can be priced by taking risk-neutral expecations of their discounted payoff.
- This can be accomplished relatively easily for the so called affine class of models.

Framework of short rate models

- Dynamics

$$
\begin{aligned}
dr(t) &= \mu(r,t)\,dt + \sigma(r,t)\,dW(t) \\
r(t) &= r \ \ (with\ r\ given)
\end{aligned}
$$

- Pricing

$$
\begin{aligned}
P(r(t);t,T) &= \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right) \\
C(r(t);t,T_1,T_2) &= \tilde{\mathbb{E}}_t\left(\max\left(P\left(r(T_1);T_1,T_2\right)-K;0\right)e^{-\int_t^{T_1} r(s)ds}\right)
\end{aligned}
$$

- The problem of consistency between model and market prices arises, because in general

$$
\begin{aligned}
P(r(t);t,T) &\neq P^{mkt}(t,T) \\
C(r(t);t,T_1,T_2) &\neq C^{mkt}(t,T_1,T_2)
\end{aligned}
$$

Pricing a zcb with short rate models

Example (1. Pricing zcb using the RN measure)

- Let us consider a pure discount bond, whose payoff is: $v(T) = 1$.
- Let us use the money market account as numeraire.
- The current price $P(t, T)$ of the zcb is therefore:

$$P(t, T) = \tilde{\mathbb{E}}_t \left(\exp\left(-\int_t^T r(s)ds \right) \times 1 \right).$$

- This is the approach used in short rate models, where a risk-neutral dynamics is assigned to the short rate

$$dr(t) = \mu(r, t)dt + \sigma(r, t)d\tilde{W}(t).$$

- Then the zcb price can be found if we know the moment generating function (MGF) of the time integral

$$I(t, T) = \int_t^T r(s)ds,$$

i.e. if we are able to compute $\tilde{\mathbb{E}}_t \left(\exp\left(-I(t, T)\right)\right)$.

* lack of consistency with market rates

Example (2. Pricing zcb using the RN measure)

- The procedure gives closed form expression for zcb prices depending on the choice of the drift and diffusion coefficient.

Table: Legend: HW: Hull & White, CIR: Cox-Ingersoll-Ross, MR-LN: mean-reverting lognormal.

Useful to identify overpriced or underpriced bond
Not options written on those bonds.

Model	Drift	Diffusion	Distr of $r(s)$	Distr of $I(t,T)$	MGF of $I(t,$
Merton	μ	σ	Gaussian	Gaussian	✓
Ho & Lee	$\mu(t)$	σ	Gaussian	Gaussian	✓
Vasicek	$\alpha(\mu - r(t))$	σ	Gaussian	Gaussian	✓
HW	$\alpha(\mu(t) - r(t))$	σ	Gaussian	Gaussian	✓
CIR	$\alpha(\mu - r(t))$	$\sigma\sqrt{r(t)}$	Non-Cent. χ^2	Unknown	✓
Dothan	$\mu r(t)$	$\sigma \times r(t)$	Lognormal	Unknown	✗
MR-LN	$\alpha(\mu - r(t))$	$\sigma \times r(t)$	Unknown	Unknown	✗

(speed of reversion → α)

- The above Table shows that only for some model (Merton, HL, Vasicek, HW and CIR) the zcb price is available in closed form. Therefore, their popularity at least in the academic literature.

extend to multi-factor short rate model
→ correlation of short rate between different maturities are not perfect

Some key issues I

- Distribution of the short rate
- Positivity of interest rates
- Analytical formula for bond prices
- Analytical formulas for bond options/cap/swaptions
- Mean reversion
- Implied volatility structures
- Correlation structure
- Suitability for Monte Carlo simulation
- Suitability for recombining lattices
- Market fit (bond prices/cap/swaptions)

Some key issues II

Example (3. Key Issues)

Table: Main facts about one-factor short rate models

Model	Distribution	Positivity	ZCB Pricing	Caplets/Swaptions	Mean-Reversion	Factors	Consistency
Merton	Gaussian	No	Yes	Yes	No	1	No
Vasicek	Gaussian	No	Yes	Yes	Yes	1	No
CIR	Non-central χ^2	Yes	Yes	Yes	Yes	1	No
Dothan	?	Yes	No	No	No	1	No
MR Log	?	Yes	No	No	Yes	1	No
MR CEV	?	Yes	No	No	Yes	1	No

Example (1. A Case Study: the Merton model)

- The sde
$$dr(t) = \mu dt + \sigma d\tilde{W}(s).$$
- The solution
$r(s) = r(t) + \mu(s-t) + \sigma \int_t^s dW(u) \sim \mathcal{N}\left(r(t) + \mu(s-t), \sigma^2(s-t)\right).$
- This means that if $\mu \neq 0$ the short rate can increase (in absolute value) without limits.
- Also, its variance growths linearly with time.
- On the other side, this model allows to perform analytical calculations.

Example (2. The distribution of $\int_t^T r(s)ds$)

- Integrate

$$\int_t^T r(s)ds = r(t)(T-t) + \mu\frac{(T-t)^2}{2} + \sigma\int_t^T\int_t^s dW(u)ds.$$

- It can be shown that

$$\int_t^T r(s)ds \sim \mathcal{N}\left(M(t,T), V(t,T)\right),$$

where $M(t,T) = r(t)(T-t) + \frac{\mu}{2}(T-t)^2$ and $V(t,T) = \frac{\sigma^2}{3}(T-t)^3$.

Example (3. Pricing zcb in the Merton model)

- Given that $\int_t^T r(s)ds \sim \mathcal{N}(M(t,T), V(t,T))$, using the moment generating function of a Gaussian r.v., it follows that the price of a zcb is given by

$$P(t,T) = \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right) = e^{-r(t)(T-t)-\frac{\mu}{2}(T-t)^2+\frac{\sigma^2}{6}(T-t)^3} \quad (1)$$

- In addition, the term structure of spot rates $R(t,T)$ is given by

$$R(t,T) = -\frac{ln(R(t,T))}{T-t} = r(t) + \frac{\mu}{2}(T-t) - \frac{\sigma^2}{6}(T-t)^2.$$

- The coefficient of $r(t)$ is equal to 1, this means that
  - The term structure of volatility is flat: all rates have the same volatility.

$$SDev_t(dR(t,T)) = SDev_t(dr(t)) = \sigma\sqrt{dt}$$

  - The correlation between changes of rates with different maturities is perfect

$$Corr_t(dR(t,T_1), dR(t,T_2)) = 1.$$

  - This correlation is 1 in all one-factor (short rate) models.

Market Data: Volatility of changes in spot rates

Table: Standard Deviation of daily changes in US yields

Tenor	2019	2010	2007	2005
1 Mo	3.15%	1.46%	16.98%	9.58%
3 Mo	2.17%	1.02%	11.21%	5.00%
6 Mo	2.09%	1.03%	7.01%	7.18%
1 Yr	2.54%	1.56%	6.19%	8.98%
2 Yr	3.82%	3.77%	6.74%	9.50%
3 Yr	3.94%	4.72%	6.58%	7.30%
5 Yr	4.08%	6.33%	6.33%	6.25%
7 Yr	4.14%	6.69%	5.90%	5.10%
10 Yr	3.96%	6.57%	5.28%	4.60%
20 Yr	3.85%	6.50%	4.72%	4.60%
30 Yr	3.77%	6.32%	4.61%	n.a.

Affine one-factor short rate models I

- In Affine models bond prices formula are available in closed form and are of the form
$$P(t,T) = e^{-A(t,T) - r(t)B(t,T)}.$$
for some smooth functions A and B, with $A(T,T) = B(T,T) = 0$ (why?).
- The above formula holds if and only if, for some continuous functions a, b, c, and d, the following restrictions on the diffusion and drift terms hold
$$\sigma^2(t) = a(t) + b(t)r(t), \text{ and } \mu(t, r(t)) = c(t) + d(t)r(t),$$
for example few examples of one-factor affine model with constant parameters are

Model	Drift	Diffusion
Merton	μ	σ
Vasicek	$\alpha(\mu - r)$	σ
CIR	$\alpha(\mu - r)$	$\sigma\sqrt{r}$

Affine one-factor short rate models II

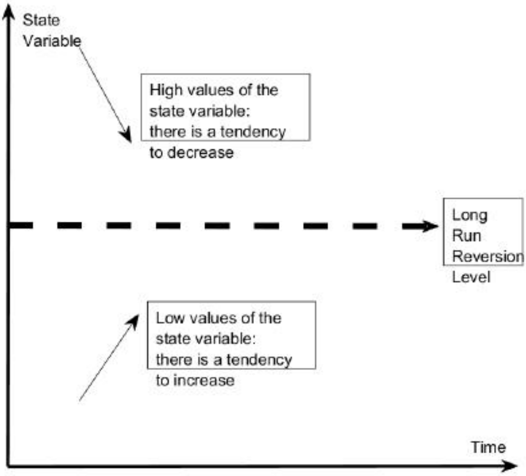

Figure: Mean reversion and expected change in the short rate

Affine one-factor short rate models

- The following Table provides the most important examples of one-factor affine models with constant parameters

Model	Dynamics dr	$B(t,T)$
Merton	$\mu dt + \sigma dW$	$T - t$
Vasicek	$\alpha(\mu - r)\, dt + \sigma dW$	$\frac{1-e^{-\alpha(T-t)}}{\alpha}$
CIR	$\alpha(\mu - r)\, dt + \sigma\sqrt{r} dW$	$\frac{2\left(e^{\phi_1(T-t)}-1\right)}{\phi_2\left(e^{\phi_1(T-t)}-1\right)+2\phi_1}$
		$\phi_1 = \sqrt{\alpha^2 + 2\sigma^2};\ \phi_2 = \phi_1 + \alpha$

- The expressions for the functions $A(t,T)$ are given in the Appendix.

Not Affine Models

- Examples of one-factor not affine models are provided in the following table

Model	Dynamics
Dothan	$dr = \lambda r dt + \sigma r dW$
MR-Lognormal	$dr = \alpha(\mu - r)\, dt + \sigma r dW$
MR-CEV	$dr = \alpha(\mu - r)\, dt + \sigma r^{\beta/2} dW.$

- For these models, no simple analytical formula for zcb prices are available and numerical methods are required.
- This fact makes their use very limited in practice.

Multi-factor extensions

- One of the main limits of the Vasicek and CIR models is the perfect correlation among changes in spot rates with different maturities.
- Several extensions have been suggested in the literature.
- Two examples are the Balduzzi, Das, Foresi and Sundaresan (BDFS) model and the Fong and Vasicek model.
- They allow for a stochastic drift and stochastic variance and they are presented in the Appendix.
- The most general class of short rate models that are analytically tractable and allow for multifactor extension belong to the affine class as discussed in Duffie and Kan (1994).

Calibrating short rate models I

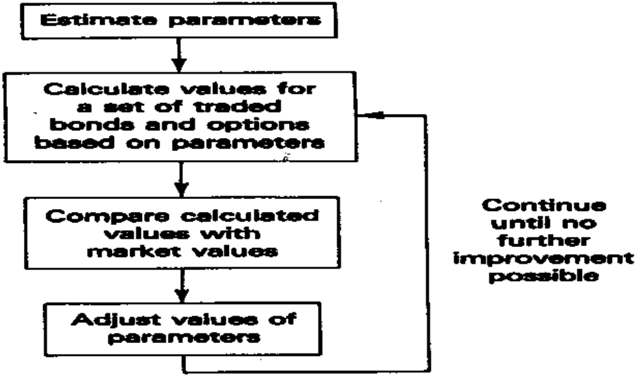

Figure: The calibration procedure

Calibrating short rate models II

- Once we have the pricing formula, we have to choose the model parameters.
- This step is called calibration: we adjust model parameters in order to make the model prices to fit market prices.
- For example, we can try to solve
  1. a minimization on the difference of weigthed squared market prices
  $$\min_{\alpha,\mu,\sigma,r} \sum_{i=1}^{n} w_i \left(P^{mkt}\left(t, T_i\right) - P^{model}\left(t, T_i; \alpha, \mu, \sigma, r\right), \right)^2$$
  2. a minimization on the difference of weigthed squared spot rates
  $$\min_{\alpha,\mu,\sigma,r} \sum_{i=1}^{n} w_i \left(R^{mkt}\left(t, T_i\right) - R^{model}\left(t, T_i; \alpha, \mu, \sigma, r\right)\right)^2 .$$
- In both cases, the weights w_i are set equal to 1 or are chosen to give greater importance to short maturities (eg $w_i = \frac{1}{T_i - t}$) or

Example (Case Study: Calibration of the Merton Model - Framework)

- In the Merton model, spot rates are linear in the model parameters (see ()), so we can recast the minimization problem as an OLS problem

$$\mathbf{R} = \mathbf{X}\beta + \epsilon$$

where $\mathbf{R}$ is $nx1$ the vector containing the observed rates for different tenors, $\mathbf{X}$ is a $nx3$ array having as first column 1, as second column the $(T_i - t)/2$ and as third column $-(T_i - t)^2/6$ and β it the vector of unknown parameters, i.e. r, μ, σ^2. ϵ is the vector of errors, given that the model is not exact.

- We can estimate β via

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{R}$$

- The estimated error is given by

$$\hat{\epsilon} = \mathbf{R} - \mathbf{X}\hat{\beta}.$$

and their estimated variance is

$$s_{\epsilon}^2 = \frac{\hat{\epsilon}'\hat{\epsilon}}{n-3}.$$

Example (Case Study. Calibration of the Merton Model: Results (1))

Excel file: FI_ShortRateModels; **Sheet**: Calibration Merton Model

Tenor $T-t$	R(t,T)	Merton	Error
0.08333	1.54%	1.54%	0.00%
0.16667	1.58%	1.54%	0.04%
0.25	1.57%	1.55%	0.02%
0.5	1.57%	1.56%	0.01%
1	1.52%	1.58%	-0.06%
2	1.62%	1.62%	0.00%
3	1.65%	1.66%	-0.01%
5	1.73%	1.75%	-0.02%
7	1.84%	1.82%	0.02%
10	1.92%	1.93%	-0.01%
20	2.21%	2.20%	0.01%
30	2.35%	2.36%	-0.01%

X		
1	0.04167	-0.001
1	0.08333	-0.005
1	0.125	-0.010
1	0.25	-0.042
1	0.5	-0.167
1	1	-0.667
1	1.5	-1.500
1	2.5	-4.167
1	3.5	-8.167
1	5	-16.667
1	10	-66.667
1	15	-150.000

$X'X$		
12	39.5	-248.0579
39.5	372.087	-3042.012
-248.06	-3042	27309

$X'R$
0.2110
0.8335
-5.5797

$(X'X)^{-1}$		
0.1816	-0.0648	-0.0056
-0.0648	0.0533	0.0053
-0.0056	0.0053	0.0006

	$\hat{\beta} = (X'X)^{-1}X'R$
$\hat{r}$	1.536%
$\hat{\mu}$	0.0895%
$\hat{\sigma}^2$	0.0000

Example (Case Study. Calibration of the Merton Model: Results (2).)

- We also have that

$$s_\epsilon^2 = 7.3824 \cdot 10^{-8}.$$

- The standard errors of the OLS estimates are

$$\mathbb{S}Dev\left(\hat{\beta}\right) = s\sqrt{diag\left(\left(\mathbf{X}'X\right)^{-1}\right)}$$

- In our numerical example, we have

	$\beta = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{R}$	s.e.	t-stat
$\hat{r}$	1.536%	0.0001	132.665
$\hat{\mu}$	0.0895%	6E-05	14.2693
$\hat{\sigma}^2$	0.0000	7E-06	5.32127
$\hat{\sigma}$	0.0059	0.00055	10.64254

Example (Case Study. Fitted Model)

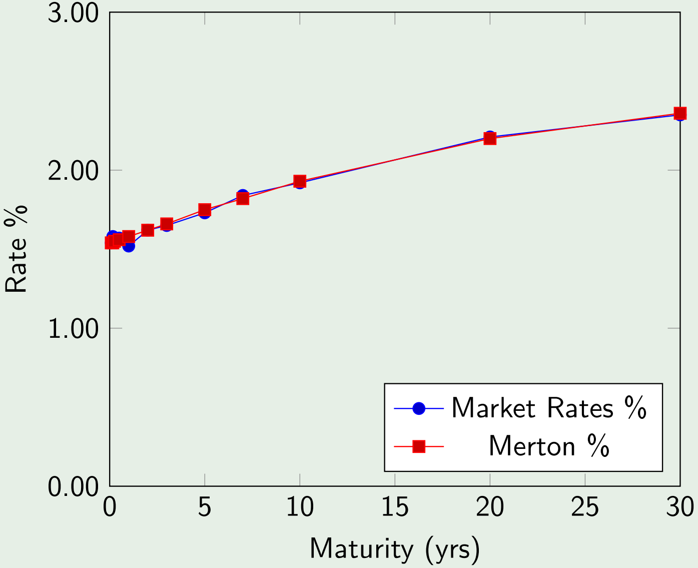

Figure: Calibrated Term Structure of Yields

Example (Case Study. Fitted Model)

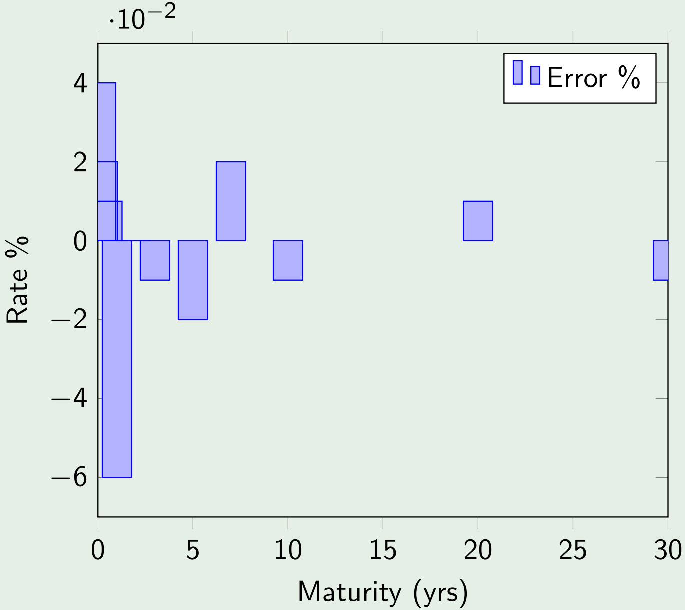

Figure: Fitted Error (%)

Calibration of the Vasicek Model

Excel file: FI_ShortRateModels; **Sheet**: Calibrating Vasicek (Euribor)

Calibration of the Vasicek model

Data: Euribor rates		Vasicek parameters	
Source: www.euribor.org		r	4.2947%
		μ	8.0856%
		α	1.9579
Trade Date	12/09/2008	σ	0.5078
Settlement Date	16/09/2008	SSE	4.615E-06

Tenor	Maturity	Adjusted Maturity	Time to Maturity	Euribor	$R^{mkt}(t,T)$	$R^{vas}(t,T)$	Pricing Error
1w	23/09/2008	23/09/2008	0.0194	4.401	4.40%	4.36%	0.03%
2w	30/09/2008	30/09/2008	0.0389	4.422	4.42%	4.43%	-0.01%
3w	07/10/2008	07/10/2008	0.0583	4.506	4.50%	4.49%	0.01%
1m	16/10/2008	16/10/2008	0.0833	4.516	4.51%	4.56%	-0.05%
2m	16/11/2008	17/11/2008	0.1722	4.764	4.74%	4.77%	-0.02%
3m	16/12/2008	16/12/2008	0.2528	4.958	4.93%	4.90%	0.02%
4m	16/01/2009	16/01/2009	0.3389	5.103	5.06%	5.01%	0.05%
5m	16/02/2009	16/02/2009	0.4250	5.147	5.09%	5.08%	0.01%
6m	16/03/2009	16/03/2009	0.5028	5.185	5.12%	5.12%	0.00%
7m	16/04/2009	16/04/2009	0.5889	5.202	5.12%	5.15%	-0.03%
8m	16/05/2009	18/05/2009	0.6778	5.229	5.14%	5.17%	-0.03%
9m	16/06/2009	16/06/2009	0.7583	5.256	5.15%	5.18%	-0.02%
10m	16/07/2009	16/07/2009	0.8417	5.285	5.17%	5.18%	-0.01%
11m	16/08/2009	17/08/2009	0.9306	5.307	5.18%	5.17%	0.01%
12m	16/09/2009	16/09/2009	1.0139	5.341	5.20%	5.17%	0.03%

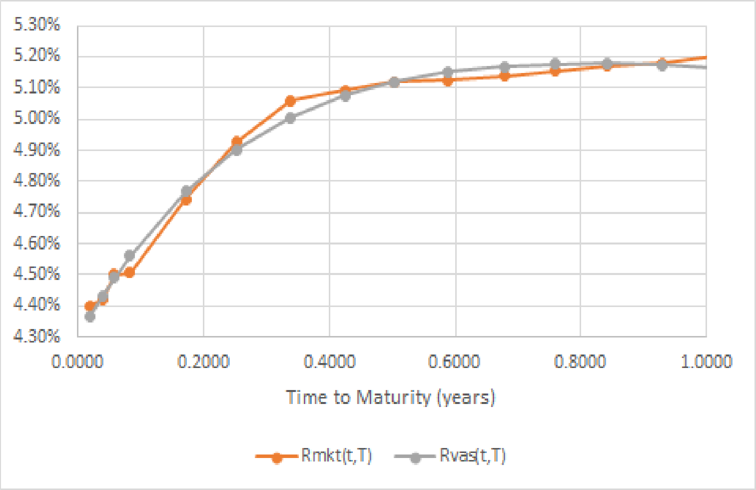

Figure: The fitted market curve using the Vasicek model

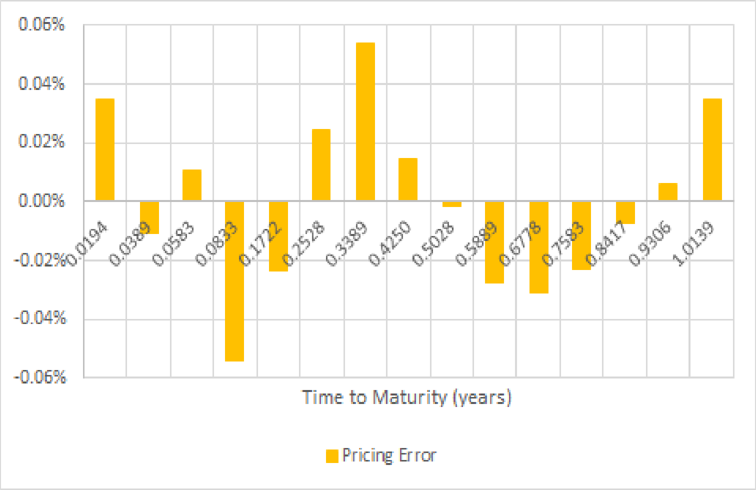

Figure: Calibrating errors using the Vasicek model: Market Rates-Model Rates

Pricing via Monte Carlo Simulation

- The analytical pricing of a zcb can be difficult for some model.
- In such a case we can use Monte Carlo simulation.
- We should proceed as follows
  1. Discretize the life of the zcb using time-intervals of size Δ, so that time s is given by
  $$s = t + j\Delta, j = 0, \cdots, N$$
  and $t + N\Delta = T$;
  2. Simulate a path i for the short rate $r^{(i)}(s)$ at the discrete times s;
  3. Given the simulated path, compute the integral of the path $\int_t^T r(s)ds$ by approximating the integral using the trapezium rule
  $$I^{(i)}(t,T) = \left(\frac{r^{(i)}(t)}{2} + \sum_{j=1}^{N-1} r^{(i)}(t + j\Delta) + \frac{r^{(i)}(t + N\Delta)}{2} \right) \Delta.$$
  4. Exponentiate and compute
  $$B^{(i)}(t,T) = e^{I^{(i)}(t,T)}.$$
  → Payoff of MMA
  5. Repeat for $i = 1, ..., M$ and average the discounted payoff of the zcb over M simulations:
  $$P^{MC}(t,T) = \frac{1}{M} \sum_{i=1}^{M} \frac{1}{B^{(i)}(t,T)}.$$

Example (Pricing zcb via Monte Carlo simulation)

- Let us suppose we have simulated 1000 paths of the short rate accordingly to our preferred interest rate model (e.g. Vasicek model) at monthly steps up to 6 months.

Sim. -Time	0	1/12	2/12	3/12	4/12	5/12	6/12
1	5%	4.4254%	3.7358%	4.3639%	4.1322%	3.8130%	3.9584%
2	5%	4.4375%	4.6937%	4.3902%	4.2765%	4.3875%	4.0319%
1000	5%	4.7326%	4.1561%	4.4195%	4.0881%	3.5899%	3.7135%

Table: Simulated Trajectories of the Vasicek model with $\alpha = 0.1$, $\mu = 4\%$, $\sigma = 1\%$, $r_0 = 5\%$.

Example ((...continued) Simulating the Money Market account)

- Then we can simulate the integral of the short rate by discretizing the integral using the trapezium rule

$$\int_u^{u+\frac{1}{12}} r(s)ds \sim \frac{r(u)+r(u+\frac{1}{12})}{2} \times \frac{1}{12},$$

and, in our example, $u = 0, 1/12, 2/12, \cdots, 5/12$.

- Therefore, we obtain simulated paths of the integral

Sim.-Time	0	1/12	2/12	3/12	4/12	5/12	6/12
Sim 1	0	0.0039	0.0073	0.0107	0.0142	0.0176	0.0208
Sim 2	0	0.0039	0.0077	**0.0115**	0.0151	0.0187	0.0223
Sim 1000	0	0.0041	0.0078	0.0113	0.0149	0.0181	0.0211

- For example the bold cell in the second simulation has been computed according to

$$I^{(2)}(0, 3/12) = I^{(2)}(0, 2/12) + \frac{1}{12} \times \frac{4.6937\% + 4.3902\%}{2} = 0.0115.$$

Example ((...continued) Estimating the zcb price)

- Simulated

Table: Integral of the short rate $I(t, T)$

Time	0	0.083333	0.166667	0.25	0.333333	0.416667	0.5
Sim 1	0	0.0039	0.0073	0.0107	0.0142	0.0176	0.0208
Sim 2	0	0.0039	0.0077	0.0115	0.0151	0.0187	0.0223
Sim 1000	0	0.0041	0.0078	0.0113	0.0149	0.0181	0.0211

Table: Paths of the money market account

Time	0	0.083333	0.166667	0.25	0.333333	0.416667	0.5
Sim 1	1	1.0039	1.0074	1.0108	1.0143	1.0177	1.0210
Sim 2	1	1.0039	1.0078	1.0116	1.0152	1.0189	1.0225
Sim 1000	1	1.0041	1.0078	1.0114	1.0150	1.0182	1.0213
$P(t, T)$	**1**	**0.9960**	**0.9924**	**0.9889**	**0.9854**	**0.9820**	**0.9788**

Example ((...continued) Estimating the zcb price)

- In the last column of the Table we have the simulated values of the integral.
- We can then compute, for each path, the simulated values of the money market account and of the zcb price

Simulation	$\int_0^{0.5} r(s)ds$	$B(0,0.5)$	$\frac{1}{B(0.5)}$
1	0.0208	1.0210	0.9794
2	0.0223	**1.0225**	0.9780
...	...	...	...
1000	0.0211	1.0213	0.9791

Table: Simulated MMA and discounted payoff

- For example the bold cell has been computed as $e^{(0.0223)} = 1.0225$ whilst the red cell according to $\frac{1}{1.0225} = 0.9780$.
- The MC estimate of the 6m zcb price is the average of the discounted payoffs:

$$\frac{0.9794 + 0.9780 + \cdots + 0.9791}{1000} = 0.9788.$$

Example ((...continued) Monte Carlo Simulation)

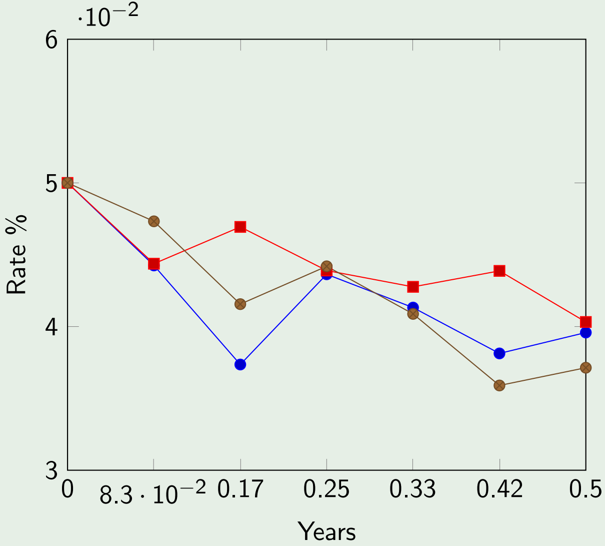

Figure: Simulated Interest Paths

Example

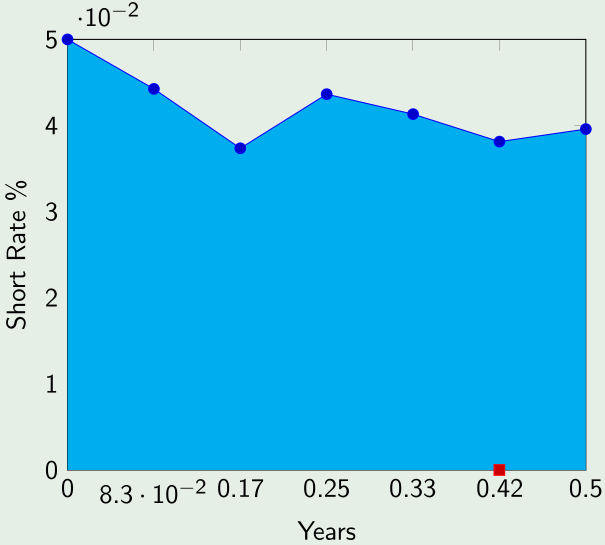

Figure: Simulated Interest Paths. The area below the simulated path represents the simulated value of the money market account.

Example ((...continued) Monte Carlo Simulation)

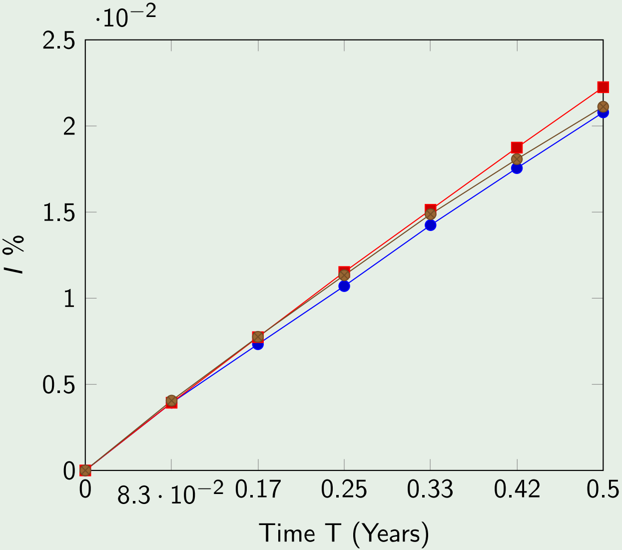

Figure: Simulated paths of $I(t, T) = \int_t^T r(s)ds, 0 = t \leq T \leq 0.5$.

Using a short rate model

- Once we have calibrate the model, we can use it to price some structured product.
- Let us consider the case of a CMS bond, i.e. a bond that pays at times $t = T_1, \cdots, T_n$ a semi-annual coupon equal to

$$\max\left(\min\left(SR; M\right)\right), m) \times 0.5 \times FV$$

where FV is the bond face value and SR is the 5-year swap rate, given by

$$SR(t) = \frac{1 - P(t, t+5)}{0.5 \times \sum_{j=1}^{10} P(t, t + \frac{j}{2})}$$

and M (m) is the maximum (minimum) coupon rate.
- We proceed as follow
  - We simulate the short rate path and the money market account;
  - At the coupon date, we use the model discount factor to obtain the discount factors needed to find the swap rate.

Pricing a bond via MC simulation I

1. Assign $r(0)$, and set Δ the swap payment dates tenor (assumed for simplicity to be equally spaced and equal to 0.5).
2. Draw a standard Gaussian random variable ϵ_j, $j = 1, \cdots, n$ and recursively simulate $r(T_j)$ on the swap payment dates according to

$$r^{(k)}(T_j) = \mu + e^{-\alpha\Delta}\left(r^{(k)}(T_{j-1}) - \mu\right) + \sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta})}\epsilon_j^{(k)},$$

   where the index k refers to the simulation, $k = 1, \cdots, M$.
3. Given the simulated short rate path, we simulate the money market account (MMA) $MMA(T) = MMA(t)e^{\int_t^T r(s)ds}$ approximating the integral using the trapezoidal rule

$$MMA^{(k)}(T_j) \cong MMA^{(k)}(T_{j-1})e^{\frac{\Delta}{2}\left(r^{(k)}(T_j)+r^{(k)}(T_{j-1})\right)},$$

   starting with $MMA(0) = 1$.

Pricing a bond via MC simulation II

4. At each future payment date s, i.e. $s = T_1, \cdots, T_n$, we simulate the discount curve according to

$$P^{(k)}(s, T_j) = e^{A(T_j - s) - B(T_j - s) r^{(k)}(s)}$$

and recalculate the swap rate according to it

$$S^{(k)}(s) = \frac{1 - P^{(k)}(s, T_n)}{0.5 \times \sum_{j=1}^{n} P^{(k)}(s, T_j)}$$

and we obtain the semi-annual coupon

$$cpn^{(k)}(s) = \max\left(\min\left(S^{(k)}(s), M\right), m\right) \times 0.5 \times FV.$$

5. The present value of the simulated coupon is

$$pv(cpn)^{(k)}(s) = \frac{cpn^{(k)}(s)}{MMA^{k}(s)}$$

Pricing a bond via MC simulation III

6. We obtain the simulated price of the bond as

$$price^k(0) = \sum_{s=1}^{n} pv(cpn)^{(k)}(s) + \frac{FV}{MMA^k(T_n)}$$

7. The bond price is obtained by averaging the simulated discounted bond payoff with respect to the number of simulations J

$$price(0) = \frac{1}{J}\sum_{k=1}^{J} price^k(0).$$

8. A concrete example is presented in the Excel file **FI_ImplementingVasicek.xlsm**, sheet:

Short rate model → exogenous short rate model

Problems with short rate models

- Short rate models recover a one-to-one relationship between instantaneous rate and zcb price.
- However, the main limit is that this relationship is obtained at a cost: zcb model prices do not coincide exactly with zcb market prices.
- Indeed, the calibration step does not allow to recover exactly market prices and some significant mispricing error is still present.
- Most traders find this unsatisfactory: how can we believe in a model for pricing derivatives, when the model itself is unable to recover the price of the underlying. Indeed a small error in the price of the underlying can lead to a large mispricing in the derivative.
- Notice that this problem does not occur in the Black-Scholes model for pricing options on stocks: the price of the stock is assumed to be given and observed on the market.
- This limit has made short rate models to be replaced by the so called `consistent models`, i.e. models that take as given zcb prices and focus the attention on the dynamics of the `instantaneous forward rates`.
- This is the class of Heath-Jarrow-Morton models.

Conclusions

- We have presented Short-rate models.
- Their great advantage is the analytical tractability that makes their implementation very easy.
- Their main limit is the lack of consistency to market quotations.
- One factor specifications also generate perfect correlation among movements at different points of the yield curve.
- This limits their use for pricing interest rate derivatives
- They are mainly used for relative value trading, i.e. for exploiting mispricing on the yield curve.
- A detailed example is given in Veronesi book, chapter 16.
- The consistency problem is solved moving to their exogenous extension.

Appendix

The Merton Model

Assumptions

- The Merton model is based on two assumptions:
  1. the expected change and the volatility of the short rate are constant;
  2. the instantaneous variations in the short rate have normal distribution.

- The two assumptions (constant expected change and constant volatility) can be formulated setting the (risk-neutral) expected change in the short rate to be equal to

$$\tilde{\mathbb{E}}_t\left(dr\right) = \mu dt,$$

where μ is a constant and is called drift, and

$$\tilde{\mathbb{V}}_t\left(dr\right) = \sigma^2 dt,$$

where σ is a positive constant and is called absolute volatility or diffusion coefficient;

- The two assumptions can be combined with the normality assumption to get the following risk-neutral dynamics to the short rate

$$dr(t) = \mu dt + \sigma dW(t), \tag{2}$$

where $dW(t)$ is the increment of the Wiener process, i.e.

$$dW\left(t\right) \sim \mathcal{N}\left(0, dt\right).$$

- Therefore
$$dr(t) \sim \mathcal{N}\left(\mu dt, \sigma^2 dt\right).$$
- This is equivalent to say that
$$r(s) = r(t) + \int_t^s \mu ds + \int_t^s \sigma dW(u),$$
or equivalently
$$r(s) = r(t) + \mu \times (s - t) + \sigma \int_t^s dW(u).$$
- In order to price a zcb we now need the distribution of $\int_t^T r(s)ds$ and then to compute $\tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right)$.

- We observe that

$$\int_t^T r(s)ds = r(t) \times (T - t) + \mu \int_t^T (s - t)ds + \sigma \int_t^T \int_t^s dW(u)ds.$$

or equivalently

$$\int_t^T r(s)ds = r(t) \times (T - t) + \mu \frac{(T - t)^2}{2} + \sigma \int_t^T \int_t^s dW(u)ds.$$

- With a change of integration, we can write

$$\int_t^T \int_t^s dW(u)ds = \int_t^T \left(\int_s^T ds \right) dW(u).$$

- In addition, we can observe that (exploiting the property of the expected value and the variance for a sum of independent terms)

$$\int_t^T \left(\int_s^T ds \right) dW(u) \sim \mathcal{N}\left(0, \int_t^T \left(\int_s^T ds \right)^2 du \right).$$

The Merton Model III

Solving the model

- Therefore

$$\int_t^T r(s)ds \sim \mathcal{N}\left(r(t)\times(T-t)+\mu\frac{(T-t)^2}{2},\sigma^2\int_t^T\left(\int_s^T ds\right)^2 du\right).$$

- We also recall the following property of the moment generating function of a Gaussian r.v.

Fact (Moment Generating Function of a normal random variable)

If X is a Gaussian random variable with mean m and variance v^2

$$\tilde{\mathbb{E}}_t\left(e^{-\lambda X}\right)=e^{-\lambda m+\frac{v^2}{2}}.$$

This is the so called moment generating function (mgf) of a normal random variable.

The pricing formula in the Merton model

Fact (Pricing formula)

Using the property of the mgf of the normal random variable and setting $X = \int_t^T r(s)ds$, $m = r(t) \times (T - t) + \mu \frac{(T-t)^2}{2}$, $v^2 = \sigma^2 \int_t^T \left(\int_s^T ds \right)^2 du = \sigma^2 \frac{(T-t)^3}{3}$ *and* $\lambda = 1$ *we have the following pricing formula for a zcb in the Merton model:*

$$P(t, T) = \tilde{\mathbb{E}}_t \left(e^{-\int_t^T r(s)ds} \right) = e^{-r(t)(T-t)-\mu \frac{(T-t)^2}{2} + \sigma^2 \frac{(T-t)^3}{6}}.$$

In addition, the spot rate is linear in r

$$R(t, T) = -\frac{\ln P(t, T)}{T - t} = r(t) + \mu \frac{(T-t)}{2} - \sigma^2 \frac{(T-t)^2}{6}.$$

The shape of the term structure

Fact (The shape of the term structure)

- *The shape of the term structure as function of the time to maturity depends on the quantity*

$$r(t) + \mu \frac{(T-t)}{2} - \sigma^2 \frac{(T-t)^2}{6}.$$

- *This a parabola, with the vertex occurring in*

$$T - t = \frac{3\mu}{2\sigma^2}.$$

- *Therefore, for $T - t > 0$ the term structure is*
  - *decreasing if $\mu \leq 0$*
  - *will be increasing and then decreasing if $\mu > 0$*

The volatility term structure in the Merton model

Fact (Volatility term structure)

- *It is defined as the standard deviation of absolute changes in the spot rates in the time unit ($dt = 1$).*

$$SDev\left(dR(t,T)\right) = SDev\left(dr(t)\right) = \sigma.$$

- *This term structure is therefore flat: spot rates with different maturities have the same volatility.*

The correlation structure in the Merton model

Fact (Volatility term structure)

- *It is defined as the correlation of absolute changes in spot rates with different maturity.*

$$Corr\left(dR(t, T_1), dR(t, T_2)\right) = Corr\left(dr(t), dr(t)\right) = 1.$$

- *This correlation is 1.*
- *This is a problem for all one-factor (short rate) models.*

Is the Merton model a good model?

- The model is analytically tractable: closed form expression for pricing zcb (and even more complex derivatives).
- The model is simple to simulate (so suitable for Monte Carlo pricing of very sophisticated financial instruments).
- Coding the pricing formula in VBA, Matlab or C does not represent a problem.
- The assumption of constant drift and volatility implies unrealistic changes in the term structure.
- The normality assumption implies that interest rates can become negative.
- Having just three parameters (r, μ and σ), the model is not able to fit exactly the term structure of zcb prices. Only three zcb's can be priced exactly.
- Being a one-factor model, changes in spot rates with different maturities are perfectly correlated, against the empirical evidence.

The Vasicek Model

Assumptions

- **Reference** Vasicek, O. (1977). An equilibrium characterization of the term structure. Journal of Financial Economics 5 177–188. Available at: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.164.447&rep=rep1&type=pdf
- The Vasicek model is based on two assumptions:
  1. the short rate shows mean reversion;
  2. the instantaneous variations in the short rate have normal distribution with constant volatility.
- Mean-reversion means to assume that the economy tends toward some equilibrium based on such fundamental factors as the productivity of capital, long-term monetary policy, and so on, short-term rates will be characterized by mean reversion.
  - When the short-term rate is above its long-run equilibrium value, the drift is negative, driving the rate down toward this long-run value.
  - When the rate is below its equilibrium value, the drift is positive, driving the rate up toward this value.

The Vasicek Model I

Mathematical Formulation

- The idea of mean-reversion can be formulated setting the (risk-neutral) expected change in the short rate to be equal to

$$\tilde{\mathbb{E}}_t\left(dr\right) = \alpha\left(\mu - r\left(t\right)\right)dt,$$

where:

a μ denotes the long-run value of short rate in the risk-neutral world;

b α denotes the speed of mean reversion, $(\alpha > 0)$.

- if $r(t) > \mu$, then

$$\tilde{\mathbb{E}}_t\left(dr\right) < 0,$$

  i.e. we expect a decrease in the short rate

- if $r(t) < \mu$, then

$$\tilde{\mathbb{E}}_t\left(dr\right) > 0,$$

  i.e. we expect an increase in the short rate.

The Vasicek Model II

Mathematical Formulation

- The normality assumption of the short rate changes coupled with the idea of mean-reversion can be formulated assigning the following risk-neutral dynamics to the short rate

$$dr(t) = \alpha \times (\mu - r(t))\, dt + \sigma \times dW(t), \tag{3}$$

where:

- σ is the diffusion coefficient, $(\sigma > 0)$;
- $dW(t)$ is the increment of the Wiener process, i.e. $dW(t) \sim \mathcal{N}(0, dt)$.
- Solving the sde, we get

$$r(s) = e^{-\alpha(s-t)} \times (r(t) - \mu) + \mu + \sigma \int_t^s e^{-\alpha(s-u)} dW(u).$$

and

$$r(s)|r(t) \sim \mathcal{N}\left(e^{-\alpha(s-t)}(r(t) - \mu) + \mu; \sigma^2 \frac{1 - e^{-2\alpha(s-t)}}{2\alpha}\right) \to \mathcal{N}\left(\mu; \frac{\sigma^2}{2\alpha}\right).$$

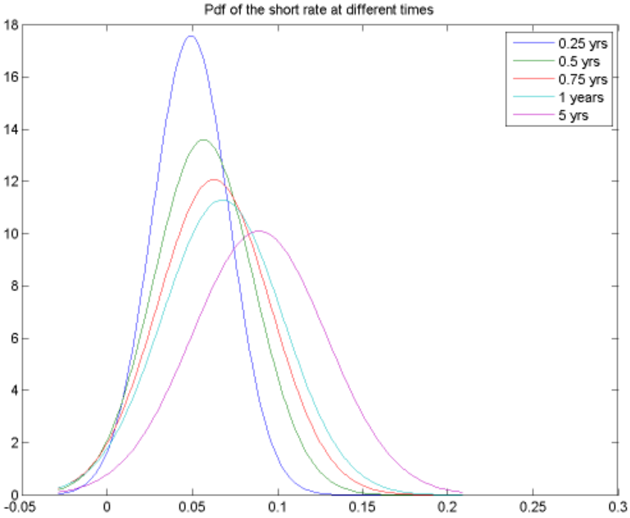

Figure: Density of the Vasicek process at different time horizons: $\alpha = 0.8, \mu = 0.09, \sigma = 0.05$

Solving the Vasicek model I

1. Compute the distribution of $\int_t^T r(s)ds$, and observe that

$$\int_t^T \int_t^s e^{-\alpha(s-u)} dW(u) ds = \int_t^T \left(\int_s^T e^{-\alpha(s-u)} ds \right) dW(u),$$

and therefore

$$\int_t^T \left(\int_s^T e^{-\alpha(s-u)} ds \right) dW(u) \sim \mathcal{N}\left(0, \int_t^T \left(\int_s^T e^{-\alpha(s-u)} ds \right)^2 du \right).$$

2. This fact allows us, exploiting the m.g.f. of a normal random variable, to compute $P(t,T) = \tilde{\mathbb{E}}_t \left(e^{-\int_t^T r(s)ds} \right)$.

Solving the Vasicek model II

3. In particular, we get

$$P(t,T) = e^{A(t,T) - B(t,T)\times r(t)},$$

where

$$\begin{aligned} B(t,T) &= \frac{1-e^{-\alpha(T-t)}}{\alpha}, \\ A(t,T) &= (B(t,T)-(T-t))\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2 B(t,T)^2}{4\alpha}. \end{aligned}$$

Non-linear Least Square Estimation

Term structure shapes in the Vasicek model I

- Given the exponential form of the zcb price, the spot rate is linear in $r(t)$

$$R(t,T) = -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t} \times r(t),$$

- The long-term spot rate is obtained by letting T to tend to $+\infty$

$$R(t,\infty) = \mu - \frac{\sigma^2}{2\alpha^2},$$

- In particular, it follows that the term structure is
  1. monotonically increasing if $r(t) < R(t,\infty) - \frac{\sigma^2}{4\alpha^2}$;
  2. humped if $R(t,\infty) - \frac{\sigma^2}{4\alpha^2} < r(t) < R(t,\infty) + \frac{\sigma^2}{2\alpha^2}$;
  3. monotonically decreasing if $r(t) > R(t,\infty) + \frac{\sigma^2}{2\alpha^2}$;

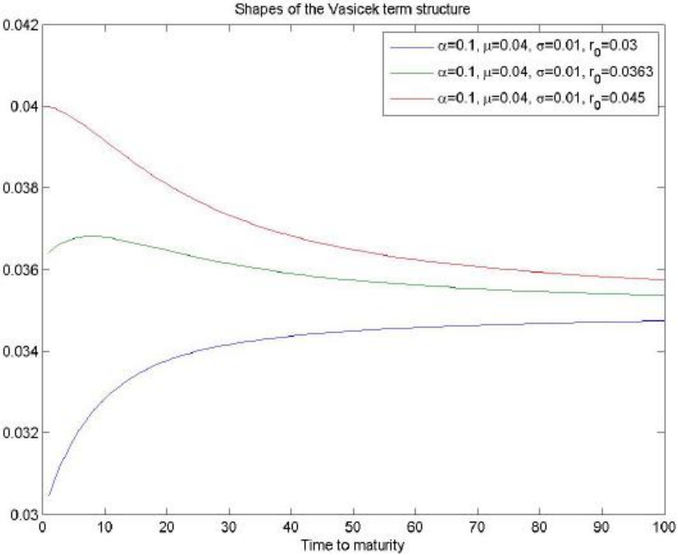

Figure: Term Structure of spot rates in the Vasicek model

The volatility term structure in the Vasicek model

- The volatility term structure is the plot of the volatility of spot rates againt time to maturity.
- Typically, the observed volatility term structure is monotonically decreasing.
- In particular, in the Vasicek model we have that, by using the Ito's lemma, the standard deviation of spot rates is

$$SDev\left(R\left(t, T\right)\right) = \sigma \frac{B\left(t, T\right)}{T - t},$$

- This is a declining function of time, provided that $\alpha > 0$.

Fact

The mean reversion property ($\alpha > 0$) allows us to generate a decreasing term structure of volatilities.

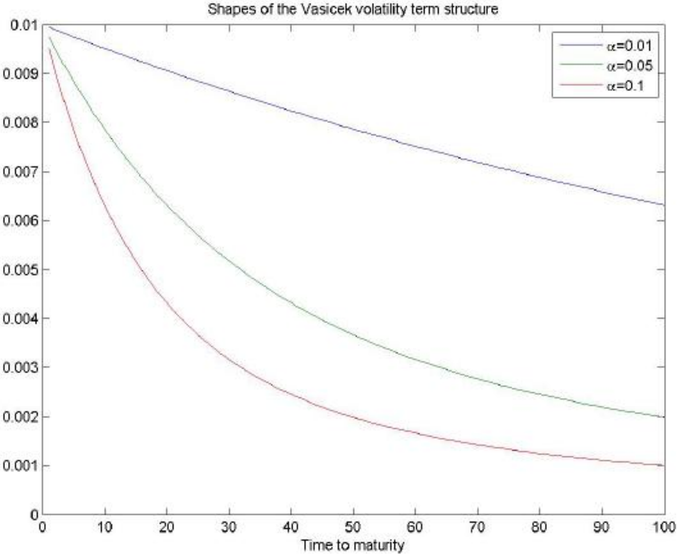

Figure: Term Structure of volatilities of spot rates in the Vasicek model

The Non-Affine Exponential Vasicek model I

- A natural way of obtaining a lognormal short rate model is by assuming that $y = \ln(r)$ follows an OU process.

$$dy_t = (\theta - ay_t)\, dt + \sigma dW_t$$

- Then the short rate has the following dynamics:

$$dr_t = r_t \left(\theta + \frac{\sigma^2}{2} - a \ln r_t\right) dt + \sigma r_t dW_t$$

- Interpretation of the model parameters:
  - a: mean-reversion speed (it measures the speed at which $\log r_t$ tends to its long-term value;
  - σ: standard deviation rate of dr_t / r_t;
  - θ / a: long-term level of the log-rate
- The process $r(t)$, being an exponential of a Gaussian r.v., is lognormally distributed.

The Non-Affine Exponential Vasicek model II

- The short rate is always mean-reverting.
- The model is not affine.
- No explicit formulas for zero coupon bonds and options on zcb are available.
- The lognormal assumption on r implies the explosion of the bank account:

$$\mathbb{E}_0\left(B\left(\Delta t\right)\right) = \mathbb{E}_0\left(e^{\int_0^{\Delta t} r(u)du}\right) = \infty$$

- In practice, this model is always applied using trees, with a finite number of states, and, therefore, finite expectations.

The Cox-Ingersoll-Ross model

Assumption

- **Reference**: COX, J., INGERSOLL, J. and ROSS, S. (1985). A theory of the term structure of interest rates. Econometrica 53 385–408. Available at `http://www.fin.ntu.edu.tw/~tzeng/course/CIR(1985)-Eca.pdf`
- The Vasicek model assumes that the volatility of the short rate is independent of the level of the short rate, i.e. $\mathbb{V}ar\left(dr\right) = \sigma^2 dt$.
- This is almost certainly not true at extreme levels of the short rate.
  - Periods of high inflation and high short-term interest rates are inherently unstable and, as a result, the basis point volatility of the short rate tends to be high.
  - Also, when the short-term rate is very low, its basis point volatility is limited by the fact that interest rates cannot decline much below zero.
- Economic arguments of this sort have led to specifying the volatility of the short rate as an increasing function of the short rate.

- The risk-neutral dynamics of the Cox-Ingersoll-Ross (CIR) model are

$$dr = \alpha \times (\mu - r(t)) \times dt + \sigma \times \sqrt{r(t)} \times dW(t),$$

so that

$$\mathbb{V}ar(dr) = \sigma^2 \times r(t) \times dt.$$

- As in the Vasicek model, the short rate features mean-reversion.
- In addition, respect to the Vasicek model, the short rate is not allowed to assume negative values.
- The model belongs to the affine class, so that the zcb prices are exponential functions of r

$$P(t, T) = e^{-B(t,T)r(t)+A(t,T)},$$

where B is given by

$$B(t, T) = \frac{2(e^{\phi_1(T-t)} - 1)}{\phi_2 \left(e^{\phi_1(T-t)} - 1\right) + 2\phi_1},$$

where $\phi_1 = \sqrt{\alpha^2 + 2\sigma^2}$; $\phi_2 = \phi_1 + \alpha$ and $A(t, T)$ is given in Brigo-Mercurio, 2006, pag. 64-66.

Term structure shapes in the CIR model I

- Given the exponential form of the zcb price, the spot rate is linear in $r(t)$

$$R(t,T) = -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t} \times r(t),$$

  and the expressions for A and
- The long-term spot rate is obtained by letting T to tend to $+\infty$

$$R(t,\infty) = \frac{2\alpha\mu}{\gamma+\alpha},$$

  where $\gamma = \sqrt{(\alpha^2 + 2\sigma^2)}$.
- In particular, it follows that the term structure is
  1. monotonically increasing if $r(t) < R(t,\infty)$;
  2. humped if $R(t,\infty) < r(t) < \mu$;
  3. monotonically decreasing if $r(t) > \mu$;

The distribution of the short rate in the CIR model I

- In order to get an intuition on the distribution of the short rate, let us consider the SDE
$$dX(t) = -\frac{\alpha}{2}X(t)dt + \frac{\sigma}{2}dW(t).$$
- We recognize that $X(t)$ is a Gaussian process with mean $m(t)$ and variance $s^2(t)$:
$$m(t) = e^{-\frac{\alpha t}{2}}X(0) \quad s^2(t) = \sigma^2\frac{(1 - e^{-\alpha t})}{8\alpha}.$$
- Now let us define $R(t) = X^2(t)$ and using the Ito's lemma we have
$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t),$$
with $\mu = \sigma^2/(4\alpha)$.
- Therefore $r(t)$ follows a square root process.

The distribution of the short rate in the CIR model II

- To understand what is the distribution of R, we proceed as follows.
  1. With a little abuse of notation, let us write
  $$X = sZ + m,$$
  where Z is a standard Gaussian random variable.
  2. It follows that r is the square of a non-standard Gaussian distribution, i.e.
  $$r = X^2 = s^2 \left(Z + \frac{m}{s}\right)^2.$$
  3. This implies that
  $$\frac{r}{s}$$
  has a non central chi-square distribution with 1 degree of freedom and parameter of non-centrality m/s (see Wikipedia).
- This shows that the distribution of r, i.e. the solution of the square-root SDE, is related to a non-central chi-square distribution with 1 degree of freedom and parameter of non centrality m/s.

The distribution of the short rate in the CIR model III

- However, if we generalize to

$$r(t) = \sum_{i}^{d} X_i^2(t)$$

where the X_i are d iid processes like in the case considered above, with coefficients α_i and σ_i, r will still have a non-central chi-square distribution but now with d degrees of freedom

- The result can be further generalized to the more general case of a non integer number d.

The distribution of the short rate in the CIR model IV

Distribution of r in the CIR model Let us assume that $r(t)$ follows the square-root SDE

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t).$$

Let us set

$$k = \frac{\sigma^2\left(1 - e^{-\alpha(T-t)}\right)}{4\alpha}.$$

Then, the distribution of

$$\frac{r(T)}{k}$$

conditioned on $r(t)$ is a non-central chi-square distribution with d degrees of freedom and non centrality parameter λ, where

$$d = \frac{4\alpha\mu}{\sigma^2}, \lambda = \frac{4\alpha r(t)}{\sigma^2(e^{\alpha(T-t)} - 1)}.$$

The distribution of the short rate in the CIR model V

In particular, the density of $r(T)$, conditioned on $r(t)$, is given by

$$\frac{1}{2k}\left(\frac{r(T)e^{\alpha(T-t)}}{r(t)}\right)^{\frac{d/2-1}{2}} e^{-\frac{r(t)e^{-\alpha(T-t)}+r(T)}{2k}} I_{d/2-1}\left(\frac{\sqrt{r(t)r(T)e^{\alpha(T-t)}}}{e^{\alpha(T-t)}k}\right),$$

where $I_v(x)$ is the modified Bessel function of the first type of order v. In addition, applying the properties of the non-central chi-square distribution, the expectation of $r(T)$ given $r(t)$ is

$$\mu_r(T) = k(d+\lambda) = \mathbb{E}_t(r(T)) = e^{-\alpha(T-t)}r(t) + \mu\left(1 - e^{-\alpha(T-t)}\right).$$

and its variance is

$$\mathbb{V}ar_t(r(T)) = 2k^2(d+2\lambda) = r(t)\left(\frac{\sigma^2}{\alpha}\right)\left(e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right) + \mu\left(\frac{\sigma^2}{2\alpha}\right)\left(1\right.$$

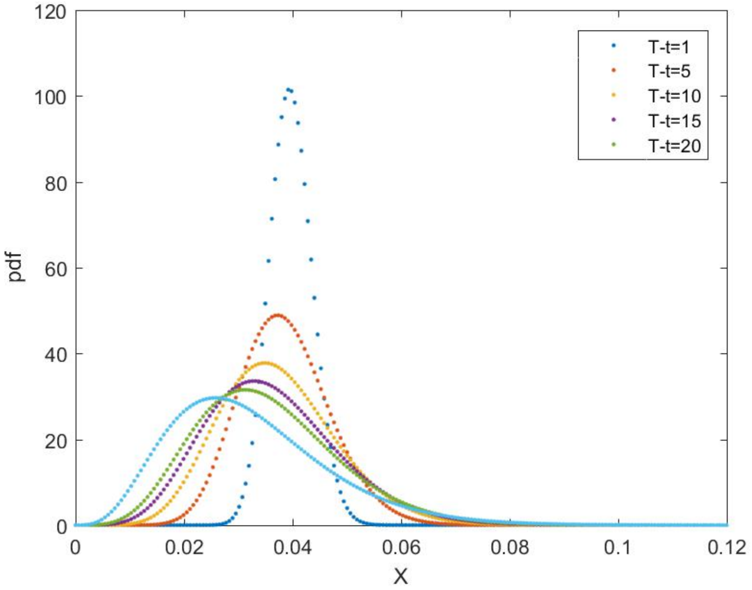

Figure: Density of the short rate in the CIR model at different horizons T. $\alpha = 0.1$, $\sigma = 0.01$, $r(t) = 0.03$, $\mu = 0.05$.

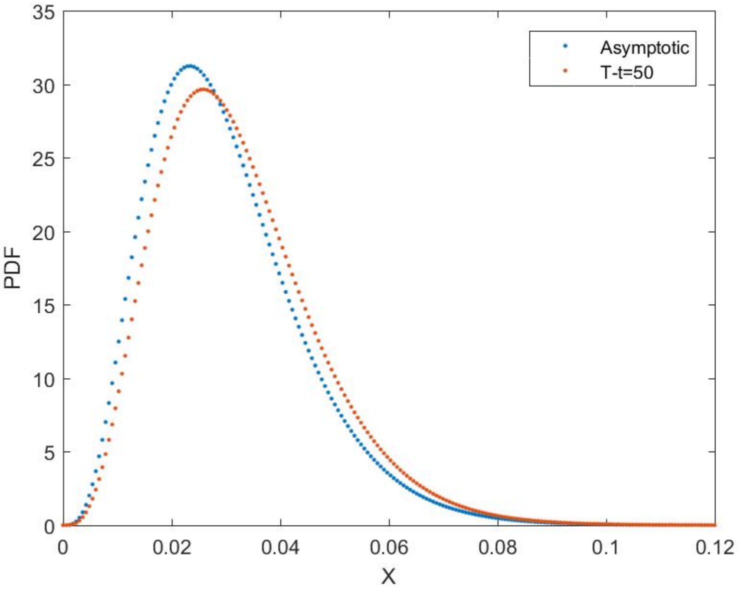

Figure: Asymptotic density in the CIR model. $\alpha = 0.03$, $\sigma = 0.02$, $r(t) = 0.04$, $\mu = 0.03$.

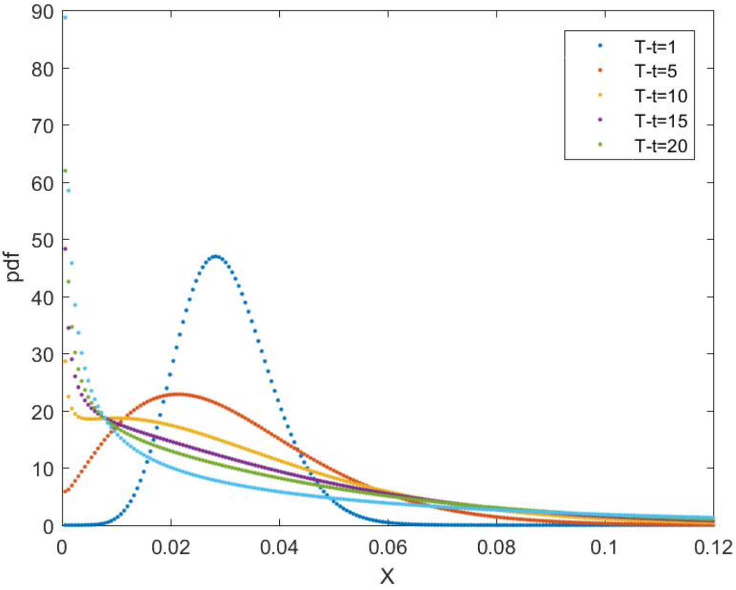

Figure: Short rate density in the CIR model when the Feller condition is violated (Feller condition guarantees that the CIR process will never reach 0 if $2\alpha\mu \geq \sigma^2$). Parameters: $\alpha = 0.01$, $\sigma = 0.05$, $r(t) = 0.03$, $\mu = 0.05$.

The BDFS model I

- **Reference**: BALDUZZI, P., DAS, S. R., FORESI, S. and SUNDARAM, R. (1996). A simple approach to three-factor affine term structure models. J. Fixed Income 6 43–53.
- The short rate dynamics in the BDFS model have the following properties
  1. The BDFS model is a multifactor model.
  2. The BDFS model allows for mean-reversion as the Vasicek and the CIR models.
  3. However, here the long run mean is not constant, but time varying according to a second sde.

The BDFS model II

4. In addition, the instantaneous volatility is not constant but changing according to a square-root process.

$$
\begin{aligned}
dr &= k\left(\mu\left(t\right) - r\left(t\right)\right)dt + \sqrt{v\left(t\right)}dW_1\left(t\right),\\
d\mu\left(t\right) &= \alpha\left(\beta - \mu\left(t\right)\right)dt + \eta dW_2\left(t\right),\\
dv\left(t\right) &= a\left(b - v\left(t\right)\right)dt + \phi\sqrt{v\left(t\right)}dW_3\left(t\right),\\
E\left(dW_2\left(t\right)dW_j\left(t\right)\right) &= 0, j = 1, 3,\\
E\left(dW_1\left(t\right)dW_3\left(t\right)\right) &= \rho_{1,3}dt
\end{aligned}
$$

- A particular case of the BDFS model is the Fong-Vasicek model where μ is assumed to be constant.

The BDFS model III

- Both models belong to the affine class, and the zcb price formula is given by

$$P(t,T) = e^{A(t,T) - B(t,T)r - C(t,T)\mu - D(t,T)v},$$

where

$$\begin{aligned} B(t,T) &= \frac{1 - e^{-k(T-t)}}{k} \\ C(t,T) &= \frac{1 - e^{-k(T-t)} + \frac{k}{\alpha} e^{-\alpha(T-t)} \left(1 - e^{-\alpha(T-t)}\right)}{\alpha - k} \end{aligned}$$

and $A(t,T)$ and $D(t,T)$ satisfy two coupled ordinary differential equations.

The BDFS model IV

- A limit of this model is related to the fact that we can hedge the volatility exposure by taking position in discount bonds (we say that volatility is spanned by the discount curve).
- In reality, there is empirical evidence that this risk cannot be hedged perfectly by trading only in discount bonds.
- Therefore, the necessity for multifactor models with unspanned stochastic volatility, such that the volatilities of discount bonds depend on state variables that are not included in the state variables used for the reconstruction of the discount curve.
- This is discussed for example in Collin-Dufresne and Goldstein (2002).

Consistent (Exogeneous) Short Rate Models

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Academic Year 2019-20

References

Useful Readings

- Brigo - Mercurio, chapter
- Veronesi book, chapters 14-15-16.
- Vasicek Tutorial

Excel Files

- FI_ShortRateModels.xls
- FI_ImplementingVasicek.xls

Outline I

1. **Exogenous short rate models**

Problems with short rate models

- Short rate models recover a one-to-one relationship between instantaneous rate and zcb price.
- However, the main limit is that this relationship is obtained at a cost: zcb model prices do not coincide exactly with zcb market prices.
- Indeed, the calibration step does not allow to recover exactly market prices and some significant mispricing error is still present.
- Most traders find this unsatisfactory: how can we believe in a model for pricing derivatives, when the model itself is unable to recover the price of the underlying. Indeed a small error in the price of the underlying can lead to a large mispricing in the derivative.
- Notice that this problem does not occur in the Black-Scholes model for pricing options on stocks: the price of the stock is assumed to be given and observed on the market.
- This limit has made short rate models to be replaced by the so called `consistent models`, i.e. models that take as given zcb prices and focus the attention on the dynamics of the `instantaneous forward rates`.
- This is the class of Heath-Jarrow-Morton models.

Exogenous short rate models

Exogenous short rate models I

The basic strategy that is used to transform an endogenous model into an exogenous short rate model is to include some time-varying parameters that allow to fit the initial term structure.

$$r(s) = F(t,s) + x(s), s \geq t$$

with $x(0) =$ and $r(t) = F(t,t)$.

- Extended Merton or Ho and Lee (discrete time version) or Jamshidian model

  $$r(s) = F(t,s) + \sigma(W_s - W_t)$$

  i.e. $dx = \sigma dW(t)$;
- Extended Vasicek (Hull-White one factor)

  $$r(s) = F(t,s) + e^{-\alpha(s-t)}x(t) + \sigma \int_t^s e^{-\alpha(s-t)} dW_s$$

  i.e. $dx = -\alpha x dt + \sigma dW(t)$;

Exogenous short rate models II

- Extended CIR (or CIR++)

$$r(s) = F(t,s) + x(s),$$

where $x(s)$ is a CIR process, i.e. $dx = \alpha(\mu - x)dt + \sigma\sqrt{x(t)}dW(t)$;

- Black-Karasinski model (BDT in discrete time)

$$r(s) = F(t,s)exp(x(s)),$$

where $x(s)$ is a mean-reverting Gaussian process, i.e.
$dx = \alpha(\mu - x)dt + \sigma dW(t)$.

Exogenous short rate models III

- The exogenous (deterministic) time-dependent shift is added to the model to guarantee consistency with an exogenously given term structure of discount factors $P^{mkt}(t, T)$.
- Indeed, if $P^{mkt}(t, T)$ are the quoted zero-coupon prices, the model-to-market consistency is guaranteed if

$$e^{-\int_t^T F(t,s)ds} = \frac{P^{mkt}(t, T)}{\tilde{\mathbb{E}}_t\left(e^{-\int_t^T x(s)ds}\right)},$$

or, $\forall T > 0$, by taking the log:

$$-\int_t^T F(t,s)ds = \ln\left(P^{mkt}(t, T)\right) - \ln\left(\tilde{\mathbb{E}}_t\left(e^{-\int_t^T x(s)ds}\right)\right)$$

Exogenous short rate models IV

- Then taking the derivative with respect to T (exploit the so called Leibnitz rule for the derivation of integrals), we have

$$
\begin{aligned}
F(t,T) = & -\frac{\partial\left(\ln P^{mkt}(t,T) - \ln \tilde{\mathbb{E}}_t\left(e^{-\int_t^T x(s)ds}\right)\right)}{\partial T} \\
= & \ f^{mkt}(t,T) + \frac{\partial \ln \tilde{\mathbb{E}}_t\left(e^{-\int_t^T x(s)ds}\right)}{\partial T}.
\end{aligned}
$$

  where $f^{mkt}(t,T)$ is the instantaneous forward rate curve.
- To improve the fitting of the volatility term structure the diffusion parameter of short rate models can be taken time-varying as well, but this has some drawbacks such as instability of calibrated parameters through time and unrealistic future volatility structures.
- Therefore this approach is not used in practice.

Exogenous short rate models V

Example (The Extended Merton Model (Ho and Lee))

- If $dx(t) = \sigma dW(s)$ and $x(0) = 0$, then

$$\tilde{\mathbb{E}}_t \left(e^{-\int_t^T x(s) ds} \right) = e^{-\frac{\sigma^2}{6}(T-t)^3}.$$

- Therefore

$$\frac{\partial \ln \tilde{\mathbb{E}}_t \left(e^{-\int_t^T x(s) ds} \right)}{\partial T} = \frac{\sigma^2}{2}(T-t)^2.$$

- So we have

$$F(t, T) = f^{mkt}(t, T) + \frac{\sigma^2}{2}(T-t)^2.$$

Exogenous short rate models VI

Table: Making short rate model consistent with the term structure

	Merton	Ho-Lee
$F(t,s)$	$r(t)(s-t)+\mu(s-t)$	$f^{mkt}(t,s)+\frac{\sigma^2}{2}(s-t)^2$
	Vasicek	Hull-White 1F
$F(t,s)$	$r(t)e^{-\alpha(s-t)}+\mu(1-e^{-\alpha(s-t)})$	$f^{mkt}(t,T)+\frac{\sigma^2}{2\alpha^2}\left(e^{-\alpha(s-t)}-1\right)^2$
	CIR	CIR++
$F(t,s)$	$r(t)$ is a CIR process	$f^{mkt}(t,T)+\frac{\partial \ln \mathbb{E}\left(e^{-\int_t^T x(s)ds}\right)}{\partial T}$
	MR-Lognormal	Black-Karasinsky
$F(t,s)$	$r(t)$ is a MR-Logn. process	Iterative procedure

Summary of exogeneous short rate models II

Model	$r > 0$	$r \sim$	ZCB	Caplets	MR	IV	ρ	MC	Trees
HL	No	Gaussian	Y	Y	No	$\sim$	1	Y	Y
HW1	No	Gaussian	Y	Y	Y	$\sim$	1	Y	Y
CIR++	Y*	Shifted NCχ^2	Y	Y	Y	$\sim$	1	Y	$\sim$
BK	Y	LN	No	No	Y	$\sim$	1	$\sim$	$\sim$

Market fit			
Model	Zcb	Caps	Swaptions
HL	Y	$\sim$	$\sim$
HW1	Y	$\sim$	$\sim$
CIR++	Y	$\sim$	$\sim$
BK	Y	$\sim$	$\sim$

Example (1. A Case Study: the Ho and Lee model)

- The short rate is given by

$$r(s) = F(t,s) + \sigma \int_t^s dW(u),$$

with $F(t,t) = r(t)$.

- Integrate

$$\int_t^T r(s)ds = \int_t^T F(t,s)ds + \sigma \int_t^T \int_t^s dW(u)ds.$$

- It can be shown that

$$\int_t^T r(s)ds \sim \mathcal{N}\left(M(t,T), V(t,T)\right),$$

where $M(t,T) = \int_t^T F(t,s)ds$ and $V(t,T) = \frac{\sigma^2}{3}(T-t)^3$.

- Using the moment generating function of a Gaussian r.v., it follows that the price of a zcb is given by

$$P_{HL}(t,T) = \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right) = e^{-\int_t^T F(t,s)ds + \frac{\sigma^2}{6}(T-t)^3} \quad (1)$$

Example (2. A Case Study: Fitting the market discount curve)

- We can now choose the function $F(t,s)$ to fit the market discount curve

$$e^{-\int_t^T F(t,s)ds+\frac{\sigma^2}{6}(T-t)^3} = P_{mkt}(t,T).$$

- Taking the log and changing sign, we have: $\int_t^T F(t,s)ds = -\ln\left(P_{mkt}(t,T)\right) + \frac{\sigma^2}{6}(T-t)^3$ and then compute the derivative with respect to T

$$F(t,T) = -\frac{\partial \ln\left(P_{mkt}(t,T)\right)}{\partial T} + \frac{\sigma^2}{2}(T-t)^2 = f(t,T) + \frac{\sigma^2}{2}(T-t)^2.$$

Example (3. A Case Study: Simulating the short rate)

1. We start at t, by setting $r(t) = f(t,t)$ and we consider the short rate at two dates ($T > s > t$, say). We have

$$r(T) = f(t,T) + \frac{\sigma^2}{2}(T-t)^2 + \sigma(W(T) - W(t)),$$

and

$$r(s) = f(t,s) + \frac{\sigma^2}{2}(s-t)^2 + \sigma(W(s) - W(t)).$$

2. We take the difference $r(T) - r(s)$ and we can write

$$r(T) = r(s) + f(t,T) - f(t,s) + \frac{\sigma^2}{2}\left((T-t)^2 - (s-t)^2\right) + \sigma(W(T) - W(s)).$$

3. Setting $T = s + ds$, we can now simulate the short rate $r(s+ds)$ step by step by using

$$r(s) + f(t,s+ds) - f(t,s) + \frac{\sigma^2}{2}\left((s+ds-t)^2 - (s-t)^2\right) + \sigma Z(s)\sqrt{ds},$$

where $Z(s)\sqrt{ds} = W(s+ds) - W(s) \sim \mathcal{N}(0, ds)$ and $Z(s) \sim \mathcal{N}(0,1)$.

A Numerical Example I

Excel: FI_ShortRateModels; **Sheet**: SimulationHoLee)

Table: A simulated path of the short rate. Parameters: $\sigma = 0.01, ds = 0.25$. For example, setting $s + ds = 1, s = 0.75$: $r(1) = 0.936\% + (0.815\% - 0.737\%) + \frac{0.01^2}{2}(1^2 - 0.75^2) + 0.01 \cdot (-0.2893) = 0.727\%$.

$s-t$	$N(0,1)$	$dW(s) = N(0,1)\sqrt{ds}$	$f(0,s)$	$r(s)$
0.00			0.500%	0.500%
0.25	0.0148	0.0074	0.579%	0.587%
0.50	0.1491	0.0746	0.658%	0.742%
0.75	0.2284	0.1142	**0.737%**	**0.936%**
1.00	-0.5785	**-0.2893**	**0.815%**	**0.727%**
1.25	-0.5225	-0.2613	0.893%	0.546%
1.50	0.2257	0.1128	0.970%	0.740%
1.75	0.5754	0.2877	1.047%	1.109%
2.00	0.7858	0.3929	1.124%	1.583%
2.25	1.9948	0.9974	1.200%	2.662%
2.50	1.6315	0.8157	1.276%	3.559%
2.75	-0.1005	-0.0502	1.351%	3.591%
3.00	-0.2496	-0.1248	1.426%	3.548%

A Numerical Example II

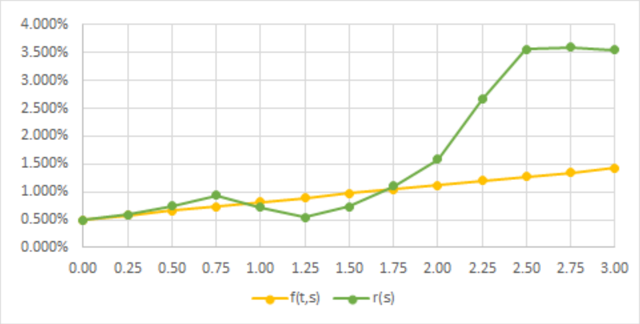

Figure: A simulated path (green line) of the short rate and the instantaneous forward curve (yellow line).

Example (4. A Case Study: Simulating the term structure)

- Given the simulated short rate at time s, we can simulate the term structure of discount factors via

$$P(s,T) = \frac{P_{mkt}(t,T)}{P_{mkt}(t,s)} e^{-(T-s)(r(s)-f(t,s))-\frac{1}{2}\cdot\sigma^2\cdot(s-t)\cdot(T-s)^2}, T > s > t.$$

- We can simulate the spot curve

$$R(s,T) = -\frac{1}{T-s}\ln\left(\frac{P_{mkt}(t,T)}{P_{mkt}(t,s)}\right) + r(s) - f(t,s) + \frac{\sigma^2}{2}(s-t)(T-s),$$

or

$$= \frac{(T-t)R(t,T)-(s-t)R(t,s)}{T-s} + r(s) - f(t,s) + \frac{\sigma^2}{2}(s-t)(T-s)$$

- Notice that: a.) in the last term, the coefficient $s-t$ is always positive, and for large s, the term structure will be positively sloped; b.) if $\sigma = 0$, then $r(s) = f(t,s), \forall s$ and the future spot rate is equal to the forward spot rate.

Table: Simulated Term Structure of Spot Rates

Time	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.5
0	**0.500%**	**0.619%**	**0.736%**	**0.853%**	**0.968%**	**1.082%**	**1.195%**	**1.306%**	**1.417%**	**1.526%**	**1.634%**
0.25	0.587%	0.862%	0.978%	1.093%	1.207%	1.319%	1.430%	1.541%	1.650%	1.758%	1.865%
0.5	0.742%	1.169%	1.284%	1.397%	1.509%	1.621%	1.731%	1.839%	1.947%	2.054%	2.159%
0.75	0.936%	1.513%	1.626%	1.738%	1.849%	1.959%	2.067%	2.175%	2.281%	2.386%	2.491%
1	0.727%	1.451%	1.563%	1.673%	1.783%	1.891%	1.998%	2.104%	2.209%	2.313%	2.416%
1.25	0.546%	1.414%	1.524%	1.633%	1.741%	1.848%	1.954%	2.059%	2.162%	2.265%	2.366%
1.5	0.740%	1.748%	1.857%	1.964%	2.071%	2.177%	2.281%	2.385%	2.487%	2.588%	2.688%
1.75	1.109%	2.254%	2.361%	2.468%	2.573%	2.677%	2.781%	2.883%	2.984%	3.084%	3.183%
2	1.583%	2.863%	2.969%	3.074%	3.178%	3.281%	3.383%	3.483%	3.583%	3.682%	3.780%
2.25	2.662%	4.073%	4.178%	4.282%	4.384%	4.486%	4.587%	4.686%	4.785%	4.882%	4.978%
2.5	3.559%	5.099%	5.203%	5.305%	5.407%	5.507%	5.606%	5.704%	5.802%	5.898%	5.993%
2.75	3.591%	5.257%	5.359%	5.460%	5.560%	5.659%	5.757%	5.854%	5.950%	6.045%	6.139%
3	3.548%	5.337%	5.438%	5.538%	5.637%	5.735%	5.831%	5.927%	6.022%	6.116%	6.209%

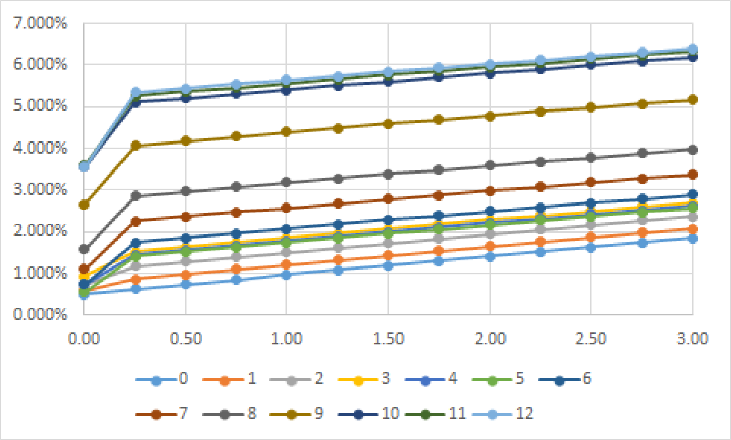

Figure: Simulated Term Structure of Spot Rates at different Time Step ($dt = 3m$).

Example (1. Case Study: Pricing a Structured Bond in the Ho and Lee model (Excel: FI_ShortRateModels; Sheet: Pricing a Bond HL))

- We need to price a 3-years bond, with semi-annual coupons (reset in advance) equal to

$$Cpn(T_i) = (1\% + \max(L(T_{i-1}, T_i) - 1\%, 0)) \cdot \alpha_{T_{i-1}, T_i}$$

and the notional at maturity, with $T_i = 0.5 \times i$, $i = 1, \cdots, 6$

- We simulate the short rate as described before and at each reset date, we compute the simulated 6-months discount factor $P^{(k)}(T_{i-1}, T_i)$ using

$$\frac{P_{mkt}(t, T_i)}{P_{mkt}(t, T_{i-1})} e^{-(T_i - T_{i-1})(r^{(k)}(T_{i-1}) - f(t, T_{i-1})) - \frac{1}{2} \cdot \sigma^2 \cdot (T_{i-1} - t) \cdot (T_i - T_{i-1})^2},$$

and then the corresponding 6-months LIBOR rate

$$L^{(k)}(T_{i-1}, T_i) = \frac{1}{\alpha_{T_{i-1}, T_i}} \left(\frac{1}{P^{(k)}(T_{i-1}, T_i)} - 1 \right), k = 1, \cdots, M,$$

and the money market account

$$MMA(T_i) = MMA(T_i - \Delta t) e^{\frac{1}{2} \cdot (r(T_i - \Delta t) + r(T_i)) \cdot \Delta t}.$$

Example (Case Study: (continued))

- At each coupon date we compute the present value of the simulated coupon

$$PV(Cpn^{(k)}(T_i)) = \frac{(1\% + max(L^{(k)}(T_{i-1}, T_i) - 1\%, 0)) \cdot \alpha_{T_{i-1},T_i}}{MMA(T_i)}$$

- We also compute the PV of the face value

$$PV(FV) = \frac{1}{MMA(T_6)}$$

- The simulated bond price is

$$BP^{(k)} = \sum_{i=1}^{6} PV(Cpn^{(k)}(T_i)) + PV(FV)$$

- The estimated bond price is

$$BP = \frac{1}{M} \sum_{k=1}^{M} BP^{(k)}.$$

Example (Case Study: (continued))

Table: A simulated path of the short rate and the corresponding coupons. E.g. $0.784\% = (1\% + max(1.569\% - 1\%, 0)) \times 0.5$ and $0.776\% = 0.784\%/1.010449$

s	$f(0,s)$	$r(s)$	$P(s, s+0,5)$	$L(s, s+0,5)$	MMA	Coupon	FV	PV(CF)
0	0.500%	0.500%	99.63%	0.738%	1			
0.25	0.579%	0.587%	99.51%	0.980%	1.001360			
0.5	0.658%	0.742%	99.36%	1.288%	1.003024	0.500%		0.498%
0.75	0.737%	0.936%	99.19%	1.633%	1.005130			
1	0.815%	0.727%	99.22%	1.569%	1.007221	0.644%		0.639%
1.25	0.893%	0.546%	99.24%	1.530%	1.008825			
1.5	0.970%	0.740%	99.08%	1.865%	1.010449	0.784%		0.776%
1.75	1.047%	1.109%	98.83%	2.375%	1.012786			
2	1.124%	1.583%	98.53%	2.991%	1.016199	0.933%		0.918%
2.25	1.200%	2.662%	97.93%	4.222%	1.021605			
2.5	1.276%	3.559%	97.43%	5.271%	1.029580	1.495%		1.452%
2.75	1.351%	3.591%	97.36%	5.431%	1.038823			
3	1.426%	3.548%	97.32%	5.512%	1.048135	2.635%	1	97.922%

Example (Case Study: (continued))

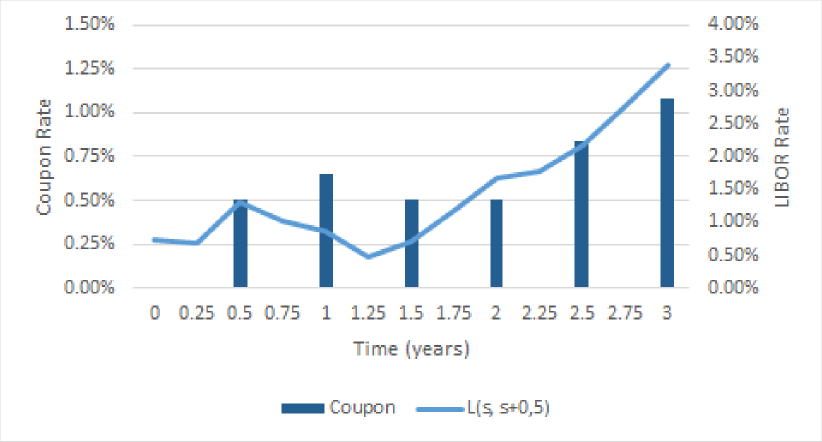

Figure: Simulated LIBOR rate and corresponding coupon rate (the rule reset in advance applies).

Example (Case Study: (continued))

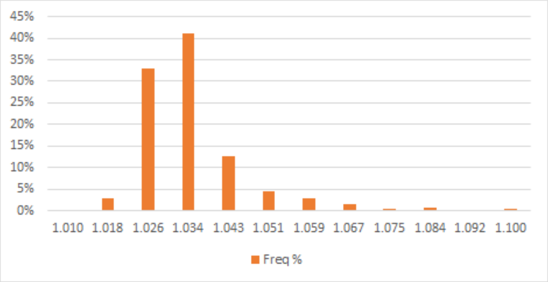

Figure: Distribution of simulated bond prices

Table: Estimated Bond Price (1000 Simulations)

	Min	Avg	Max	s.e.
PV(Cpn)	3.004%	5.952%	12.672%	0.068%
PV(FV)	88.454%	97.141%	106.909%	0.095%
PV(Bond)	101.001%	103.093%	110.003%	0.034%

Conclusions

- We have presented an extension of short-rate models.
- The extension allows us to obtain consistency to market quotations.
- This property allows their use for pricing interest rate derivatives

The Most Important Formula and its application to pricing FRA, FRN & IRS (single curve approach)

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SMM269 Fixed Income - Academic Year 2019-20

References

Useful Readings

- Any Standard Fixed Income Book.

Excel Files

- FI_Swap.xls

Outline I

1. **Replicating a floating coupon**
2. **Again the most important formula**
3. **Applications of the Fundamental Formula**
4. **Take Aways**
5. **Application 1:**
   Forward Rate Agreements
6. **Application 2:**
   FRA rates and discount factors
7. **Application 3:**
   Floating Rate Notes
8. **Application 4:**
   Interest Rate Swaps
   - A Case Study

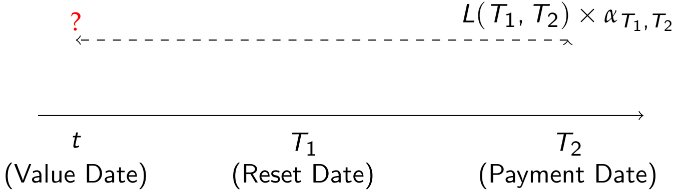
The Most Important Formula: Replicating and Pricing a Floating Amount

Question

Let us consider the amount that resets in T_1 and is paid in T_2 (reset in advance and pay in arrears)

$$Payoff(T_2) = L(T_1, T_2) \times \alpha_{T_1,T_2}$$

What is t-price of the above (random) payoff?



Replicating a floating payment I

Consider the following trading strategies in two portfolios, A and B:

- Portfolio A: floating coupon

	t	T_1	T_2
Cash flow	$-V_A = ?$	$-$	$\widetilde{L}(T_1, T_2) \times \alpha_{T_1,T_2}$

- Porfolio B: replicating portfolio
  - buy a $T_1 - zcb$, and when it expires reinvest the unit face value 1, at the prevailing Libor rate, from T_1 to T_2
  - sell a $T_2 - zcb$

	t	T_1	T_2
1)	$-P(t, T_1)$	1	$-$
	$-$	-1	$1 + \widetilde{L}(T_1, T_2) \times \alpha_{T_1,T_2}$
2)	$P(t, T_2)$	$-$	-1
	$-[P(t, T_1) - P(t, T_2)]$	$-$	$\widetilde{L}(T_1, T_2) \times \alpha_{T_1,T_2}$

Replicating a floating payment II

- By no arbitrage, $V_A(t) = V_B(t)$:

$$V_A(t) = P(t, T_1) - P(t, T_2) \tag{1}$$

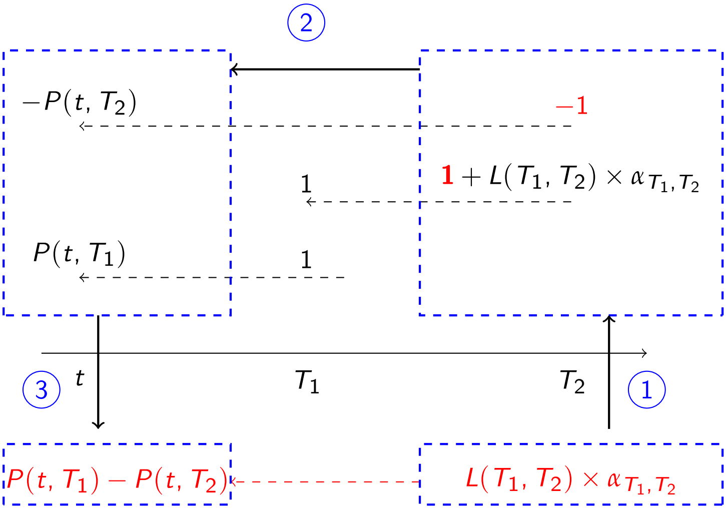
  - the current value of a floating payment is equal to the difference of two zero-coupon bonds.

- Collecting $P(t, T_2)$, we get:

$$\begin{aligned} V_A(t) &= P(t, T_1) - P(t, T_2) \\ &= P(t, T_2) \times \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right) \times \frac{\alpha_{T_1, T_2}}{\alpha_{T_1, T_2}} \\ &= P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2}. \end{aligned}$$

- The current value of a floating payment can be determined as follows:
  1. replace the unknown future Libor rate $\widetilde{L}(T_1, T_2)$ with the current simple forward rate $F(t, T_1, T_2)$;
  2. and, discount it using the riskless discount factor.

The idea behind formula (1)



Fact (Fundamental Receipe for Pricing Floating Payments)

Consider the the floating amount

$$V_{FL}(T_2) = N \times L(T_1, T_2) \times \alpha_{T_1,T_2}$$

that resets in T_1 and is paid in T_2.
Its t-fair value can be computed in two ways:

1. *as difference of two zero-coupon bonds:*

$$\boxed{V_{FL}(t) = (P(t, T_1) - P(t, T_2)) \times N.}$$

2. *via forward rate representation:*

$$\boxed{V_{FL}(t) = \underbrace{N}_{\text{notional}} \times \underbrace{F(t, T_1, T_2)}_{\text{forward rate}} \times \underbrace{\alpha_{T_1,T_2}}_{\text{accrual factor}} \times \underbrace{P(t, T_2)}_{T_2\text{-discount factor}}.}$$

Example (Using the formula)

- We have the following market quotes

T_i	$P(0, T_i)$
0.5	0.98
1	0.94

- What's the value today of a 6x12 floating payment at Libor, on a notional of Euro $1,000,000$?

	0	$T_1 = 0.5$	$T_2 = 1$
Cash flow	???	$-$	$N \times \widetilde{L}(0.5, 1) \times 0.5$

Example

- **Method 1**: via difference of two zero-coupon bond prices

✓

used for pricing

$$\begin{aligned} V_{FL}(0) &= N \times [P(0, T_1) - P(0, T_2)] \\ &= 1,000,000 \times [0.98 - 0.94] \\ &= 40,000 \end{aligned}$$

- **Method 2**: via forward rate

$$\begin{aligned} V_{FL}(0) &= N \times F(0, T_1, T_2) \times \alpha_{(T_1, T_2)} \times P(0, T_2) \\ &= 1,000,000 \times F(0, 0.5, 1) \times 0.5 \times P(0, 1) \\ &= 1,000,000 \times \frac{1}{0.5}\left(\frac{0.98}{0.94} - 1\right) \times 0.5 \times 0.94 \\ &= 1,000,000 \times 8.5106\% \times 0.5 \times 0.94 \\ &= 40,000 \end{aligned}$$

Applications of the formula

1. Pricing Forward Rate Agreement
2. Pricing Floating Rate Notes
3. Pricing Swaps
4. Pricing Floating Rate Mortgages
5. In general, pricing any kind od product having cash flows tied to a LIBOR rate.

Assumptions behind the formula

1. No counterparty risk. In particular, we need to assume that credit risk is the same for all banks and stable over time.
2. The tenor of the reference rate is the same as the tenor of the deposit.
3. The convention *resets in advance, pay in arrears* is adopted.

Assumption 1: No Counterparty Credit Risk I

- If the counterparty can default, we need to adjust for the expected losses due to her default.
- The value of the defaultable contract is then

Risk-Free Value - Present value of the expected loss.

- Assume, for aim of simplicity, that

a. the default can only occurr at expiry;

b. on default, we can recover a fraction R of the (risk-free) market value of the contract.

c. So the actual **loss given default** is

$$(1 - R) \times L(T_1, T_2) \times \alpha_{T_1, T_2}.$$

Recovery Rate

Assumption 1: No Counterparty Credit Risk II

- Let $Q(t,T)$ be the survival probability of the counterparty, i.e.

a. $Q(t,T)$ is the probability that being alive at time t she will be still alive at time T;

b. Therefore $1-Q(t,T)$ is the so called default probability, i.e. the probability of defaulting between t and T.

- The risk-free value of the contract is clearly

$$RF(t) = P(t,T_2) \times F(t,T_1,T_2) \times \alpha_{T_1,T_2}.$$

- The present value of the expected loss is called **credit value adjustment** (CVA) and is computed according to the following expression

$$CVA(t) = (1-Q(t,T_2)) \times (1-R) \times P(t,T_2) \times F(t,T_1,T_2) \times \alpha_{T_1,T_2}.$$

- By difference between risk-free value and CVA of the contract we obtain the value of the defaultable contract

* Assume no WWR
there is not dependency between r and PD

Assumption 1: No Counterparty Credit Risk III

- We can rewrite it as

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2} \times (1 - (1 - Q(t, T_2)) \times (1 - R)),$$

or, introducing an adjusted discount factor

$$P^*(t, T_2) = P(t, T_2) \times (1 - (1 - Q(t, T_2)) \times (1 - R)),$$

the value of the defaultable contract becomes

$$P^*(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2},$$

that looks very similar to the risk-free version.

Assumption 1: No Counterparty Credit Risk IV

1. Under counterparty risk, the method based on forward rates should be preferred, but using an appropriate discount factor, i.e. P^* instead of P.
2. P^* is obtained by correcting the risk-free discount factor by a quantity related to the probability of default and to the loss given default, i.e. $1 - R$.
3. Risk-adjusted discount factors are then used for discounting.
4. The above result holds under the assumption of independence between interest rates and default.

Example (Pricing a defaultable contract)

We want to price a 6x9 defaultable contract, with face value of 1000 USD.

- The risk-free discount factors for 6m and 9m are 0.98 and 0.97. The 9m survival probability of the issuer is 0.99. The recovery ratio is 0.4. The 6x9 forward rate is $(0.98/0.97 - 1)/0.25 = 4.12\%$

LGD = 0.6

- The risk-free value of the contract is

$$\mathbf{0.97} \times 4.12\% \times 0.25 \times 1000 = (0.98 - 0.97) \times 1000 = 10.$$

- The CVA of the contract is

$$0.97 \times (1 - 0.99) \times (1 - 0.4) \times 4.12\% \times 0.25 \times 1000 = 0.06.$$

- The defaultable value of the contract is $10 - 0.06 = 9.94$.
- We can also compute an adjusted discount factor

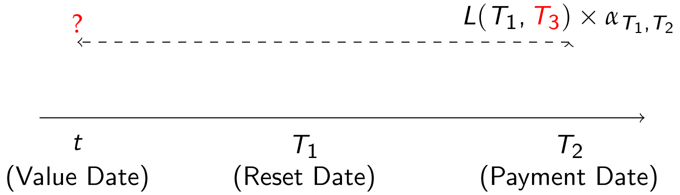
$$P^* = 0.97 \times (1 - (1 - 0.99) \times 0.6) = 0.96418,$$

so that the value of the defaultable contract is

$$\mathbf{0.96418} \times 4.12\% \times 0.25 \times 1000 = 9.94.$$

Assumption 2: LIBOR tenor vs contract tenor

- If the contract resets at time T_1 and pays at time T_2, but the underlying is a LIBOR with different tenor, $L(T_1, T_3)$ for example, the formula does not hold anymore



- The point is that $1 + L(T_1, T_3) \times \alpha_{T_1,T_2}$ is not worth anymore 1 in T_1, but

$$\frac{1 + L(T_1, T_3) \times \alpha_{T_1,T_2}}{1 + L(T_1, T_2) \times \alpha_{T_1,T_2}}.$$

- Therefore, we cannot discount it back to t: as seen from t it is a random amount.

Assumption 3: Payment and Reset in Arrears I

Reset in Advance and Pay in Arrears

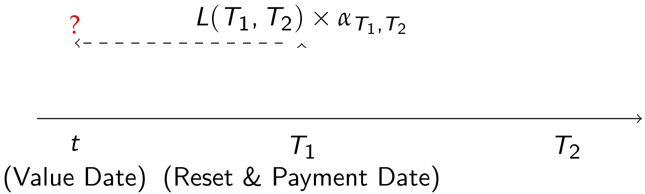
T_1 Reset Date	$\overset{\alpha_{T_1,T_2}}{\longleftrightarrow}$	T_2 Payment Date
$\downarrow$		$\downarrow$
$L(T_1, T_2) = 3\%$	$\rightarrow$	$Coupon(T_2) = 3\% \times \alpha_{T_1,T_2} \times N$

Reset and Pay in Arrears

T_1 Reset & Payment Date	$\overset{\alpha_{T_1,T_2}}{\longleftrightarrow}$	T_2 Next Reset & Payment Date
$\downarrow$		$\downarrow$
$L(T_1, T_2) = 3\%$		$L(T_2, T_3) = 2.8\%$
$\downarrow$		$\downarrow$
$Coupon(T_1) = 3\% \times \alpha_{T_1,T_2} \times N$		$Coupon(T_2) = 2.8\% \times \alpha_{T_2,T_3} \times N$

Assumption 3: Payment and Reset in Arrears II

- If the contract resets and pays at time T_1, the fundamental formula does not hold anymore.
- The cash flows are as in the figure.



- The point is that $L(T_1, T_2) \times \alpha_{T_1,T_2}$ is a random amount as seen from time t.
- Therefore, we cannot simply discount it back to t.

A Question

A Question I

- The most important formula says that the price of the floating amount

$$V(T_2) = L(T_1, T_2) \times \alpha_{T_1,T_2}$$

that resets in T_1 and is paid in T_2 is

$$V(t) = \underbrace{F(t, T_1, T_2)}_{\text{forward rate}} \times \underbrace{\alpha(T_1, T_2)}_{\text{accrual factor}} \times \underbrace{P(t, T_2)}_{T_2- \text{ discount factor}} .$$

- Is this formula consistent with pricing via expected discounted payoffs?

$$\mathbb{E}_t \left[\frac{L(T_1, T_2)}{MMA(T_2)} \right] \times \alpha_{T_1,T_2}$$

where

$$MMA(T) = e^{\int_t^T r(s) ds}$$

is the T value of a unit of money invested in t.

A Question II

The Question

So the question is to show why the following equality

$$\mathbb{E}_t\left[\frac{L(T_1, T_2) \times \alpha_{T_1,T_2}}{e^{\int_t^{T_2} r(s)ds}}\right] = P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1,T_2}$$

holds.

Using the Most Important Formula

Question

Problem (Fair Value of a spot starting floating leg)

We receive every three months, up to 12 months, an amount depending on the 3 months LIBOR rate that resets 3 months earlier, i.e. at time T_i we have

$$L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i},$$

and $T_i - T_{i-1} = 0.25$.
Determine the fair market value of this amount, given the following term structure of discount factors

Months	*0*	*3*	*6*	*9*	*12*
$P(t, T)$	*1*	*0.99*	*0.985*	*0.982*	*0.98*

Answer

Solution (The contract cash flows)

We have the following cash flows

Table: *Cash Flows of the contract*

Reset T_{i-1}	Payment T_i	Cash Flow	Fair Value
0	0.25	$L(0, 0.25) \times 0.25$	?
0.25	0.5	$L(0.25, 0.5) \times 0.25$	?
0.5	0.75	$L(0.5, 0.75) \times 0.25$	?
0.75	1	$L(0.75, 1) \times 0.25$	?

Solution (Fair value of each cash flow)

Using the zcb formula, the fair value of the amount

$$L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i}$$

is

$$FV = P(t, T_{i-1}) - P(t, T_i).$$

Using the discount term structure, we can fill in the previous Table

Table: *Cash Flows of the contract*

T_{i-1}	T_i	$P(t, T_i)$	Cash Flow	Fair Value
0	0.25	0.99	$L(0, 0.25) \times 0.25$	$1 - 0.99$
0.25	0.5	0.985	$L(0.25, 0.5) \times 0.25$	$0.99 - 0.985$
0.5	0.75	0.982	$L(0.5, 0.75) \times 0.25$	$0.985 - 0.982$
0.75	1	0.98	$L(0.75, 1) \times 0.25$	$0.982 - 0.98$
			Fair Value	*1-0.98*

Problem (Fair Value of a forward starting floating leg)

Let us consider the same problem as before, but the first reset date is not at the initial time, but forward in time, say in T_0.
In this case, we have a forward starting floating leg. Its value is obtained by filling the Table cash flows

Table: *Cash Flows of the contract*

T_{i-1}	T_i	$P(t, T_i)$	Cash Flow	Fair Value
0	0.25	0.99	No Cash flow	
0.25	0.5	0.985	$L(0.25, 0.5) \times 0.25$	$0.99 - 0.985$
0.5	0.75	0.982	$L(0.5, 0.75) \times 0.25$	$0.985 - 0.982$
0.75	1	0.98	$L(0.75, 1) \times 0.25$	$0.982 - 0.98$
			Fair Value	0.99-0.98

Question

Fact (Pricing of a forward starting floating leg)

We have the following stream of cash flows

forward starting

Table: *Cash Flows of the contract*

T_{i-1}	T_i	$P(t, T_i)$	Cash Flow	Fair Value
0	T_0	0.99	No Cash Flow	
T_0	T_1	0.985	$L(T_0, T_1) \times \alpha$	$0.99 - 0.985$
T_1	T_2	0.982	$L(T_1, T_2) \times \alpha$	$0.985 - 0.982$
T_2	T_3	0.98	$L(T_2, T_3) \times \alpha$	$0.982 - 0.98$
			Fair Value	0.99-0.98

Fact (Fair Value of a forward starting floating leg)

Let us consider the cash flows as in Table below, where

- t *is the value date;*
- T_0 *is the first reset date,* $T_0 > t$*;*
- T_1 *is the first payment date;*
- *The cash flow received in* T_i *resets in* T_{i-1}*;*
- *Last payment occurs in* T_n *and resets in* T_{n-1}*.*

T_{i-1}	T_i	$P(t, T_i)$	*Cash Flow*	*Fair Value*
t	T_0	$P(t, T_0)$	*No Cash Flow*	
T_0	T_1	$P(t, T_1)$	$L(T_0, T_1) \times \alpha_{T_0, T_1}$	$P(t, T_0) - P(t, T_1)$
T_1	T_2	$P(t, T_2)$	$L(T_1, T_2) \times \alpha_{T_1, T_2}$	$P(t, T_1) - P(t, T_2)$
...	...	...	...	...
T_{i-1}	T_i	$P(t, T_i)$	$L(T_{i-1}, T_i) \times \alpha_{T_{i-1}, T_i}$	$P(t, T_{i-1}) - P(t, T_i)$
...	...	...	...	...
T_{n-1}	T_n	$P(t, T_n)$	$L(T_{n-1}, T_n) \times \alpha_{T_{n-1}, T_n}$	$P(t, T_{n-1}) - P(t, T_n)$
			Fair Value	$P(t, T_0) - P(t, T_n)$

Table: *Cash Flows of the contract*

Take Away 1

We can recall the main results coming from the most important formula

ZCB version

- **[Single Cash Flow]** The cash flow $L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i}$ received in T_i, at time t has a fair value of

$$P(t, T_{i-1}) - P(t, T_i).$$

- **[Floating Leg]** If we receive the sequence of cash flows $L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i}$ at times $T_i, i = 1, \cdots, n$, this floating leg at time t has a fair value of

$$\sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i)) = P(t, T_0) - P(t, T_n).$$

Take Away 2

We can recall the main results coming from the most important formula
Forward Rate version

- **[Single Cash Flow]** The cash flow $L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i}$ received in T_i, at time t has a fair value of

$$P(t, T_i) \times F\left(t, T_{i-1}, T_i\right) \times \alpha_{T_{i-1},T_i}.$$

- **[Floating Leg]** If we receive the sequence of cash flows $L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i}$ at times $T_i, i = 1, \cdots, n$, this floating leg at time t has a fair value of

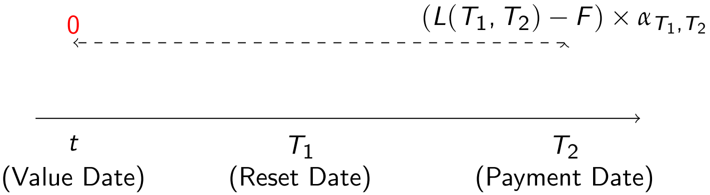
$$\sum_{i=1}^{n} P(t, T_i) \times F\left(t, T_{i-1}, T_i\right) \times \alpha_{T_{i-1},T_i}.$$

Application 1
Pricing Forward Rate Agreements

Application 1: Forward Rate Agreement I

A FRA has the cash flow as in figure below.

- At inception t, there is no cash flow.
- At time T_2, the long side (buyer) receives the floating rate and pays the fixed rate F (FRA Rate).



- The fixed rate F is called FRA rate or simple forward LIBOR rate.

Pricing on-the-run FRA

- **Problem**: how do you set F so that the contract has a zero cost?
- By applying the forward rate version of the formula, we see that the value of the contract at time t is

$$P(t, T_2) \times (F(t, T_1, T_2) - F) \times \alpha_{T_1, T_2}.$$

- Therefore, at the contract inception the FRA rate must be equal to the forward rate

$$F = F(t, T_1, T_2).$$

Pricing an off-the-run FRA

- In general, after inception, the contract value can be positive or negative, depending on the evolution of interest rates.
- If the current forward rate has increased (declined) since inception, the value of a long FRA will be positive (negative).
- The fixed rate has been fixed at the inception date t.
- We intend to value the contract at time $s, s > t$.
- Again, we apply the forward rate version of the important formula

$$FV_s(\text{Long FRA}) = P(s, T_2) \times (F(s, T_1, T_2) - F(t, T_1, T_2)) \times \alpha_{T_1, T_2}.$$

Example (Pricing the on-the-run FRA)

- On February 1, 2007 (today = time 0), we enter into a short 6×9 FRA on a notional $N = 1,000,000$ Euro.
  - Compute the FRA rate.
  - Compute the value of the contract after 3 months.

- FRA rate:

EURIBOR - Fonte: www.euribor.org

	1w	2w	3w	1m	2m	3m	4m	5m	6m	7m	8m	9m	10m	11m	12m
1-Feb-07	3.587	3.593	3.603	3.609	3.711	3.785	3.829	3.875	3.923	3.955	3.985	4.016	4.037	4.061	4.080

  - Relevant dates and rates:
    - August 2007: 6m Euribor = 3.923%
    - November 2007: 9m Euribor = 4.016%

$$F(0, 6m, 9m) = \frac{1}{0.25}\left(\frac{\frac{1}{1+3.923\%\times 0.5}}{\frac{1}{1+4.016\%\times 0.75}} - 1\right) = 4.12\%$$

The value of the short FRA is 0.

Example (Pricing the off-the-run FRA)

- Value of a short FRA after 3 months (i.e. on **May 2007**):

EURIBOR - Fonte: www.euribor.org

	1w	2w	3w	1m	2m	3m	4m	5m	6m	7m	8m	9m	10m	11m	12m
2-May-07	3.848	3.848	3.852	3.863	3.946	4.023	4.063	4.103	4.143	4.167	4.204	4.234	4.253	4.287	4.306

  - Relevant dates and rates:
    - August 2007: 3m Euribor = 4.023%
    - November 2007: 6m Euribor = 4.143%

We compute the new forward rate

$$F_{NEW} = \frac{1}{0.25}\left(\frac{\frac{1}{1+4.023\%\times 0.25}}{\frac{1}{1+4.143\%\times 0.5}} - 1\right) = 4.22\%$$

Then the value of the seasoned short FRA is

$$FRA_{SHORT} = 1ml \times 0.25 \times \frac{(4.12\% - 4.22\%)}{\frac{1}{1+4.143\%\times 0.5}} = -243.428 EUR$$

Market Quotes

	Euro		USD		Yen	
	Bid	Ask	Bid	Ask	Bid	Ask
3x6	2.22	2.24	1.29	1.33	0.07	0.11
6x9	2.41	2.42	1.57	1.61	0.1	0.14
9x12	2.64	2.66	1.94	1.98	0.15	0.19
6x12	2.55	2.56	1.76	1.8	0.13	0.17
12x18	3.05	3.07	2.59	2.63	0.29	0.33

Table: FRA Rate Quotations, October 29, 2003. Source: Il Sole 24 Ore

- Market quotes directly forward rates.
- Here 3x6 means: starts in 3 months and ends in 6 months.
- Let us consider the Euro currency:
  - we can lend N in 3 months time and in 6 months time we will receive back the amount $N \times (1 + 0.022 \times \frac{3}{12})$.
  - we can borrow N in 3 months time and in 6 months time we will pay back the amount $N \times (1 + 0.024 \times \frac{3}{12})$.

Building the discount curve using FRA's I

- The FRA rates give information relative to the discount curve.
- We observe that the present value in t of 1 unit of money that will be received in T_2 can be computed in two ways:
  - We compute the present value directly up to initial time using the spot discount factor $P(t, T_2)$

$$1 \rightarrow \underbrace{P\left(t, T_2\right)}_{\text{p.v. from } \mathrm{T}_2 \text{ to } t.}.$$

  - We compute the present value up to T_1, using the FRA rate and then the present value to the current time using the spot discount factor $P\left(t, T_1\right)$

$$1 \rightarrow \underbrace{\frac{1}{1+F\left(t, T_1, T_2\right)\left(\alpha\left(T_1, T_2\right)\right)}}_{\text{p.v. from } \mathrm{T}_2 \text{ to } \mathrm{T}_1} \rightarrow \underbrace{\frac{1}{1+F\left(t, T_1, T_2\right)\alpha\left(T_1, T_2\right)} P\left(t, T_1\right).}_{\text{p.v. from } \mathrm{T}_1 \text{ to } t}$$

Building the discount curve using FRA's II

No-arbitrage implies that

$$P(t, T_2) = P(t, T_1) \times \frac{1}{1 + F(t, T_1, T_2)\,\alpha(T_1, T_2)}. \tag{2}$$

Therefore using $P(t, T_1)$ and $F(t, T_1, T_2)$ we can build $P(t, T_2)$.

- Notice that the shortest discount factor can in general be recovered from the LIBOR rate

$$P(t, T_1) = \frac{1}{1 + L(t, T_1)\,\alpha(t, T_1)},$$

and

$$F(t, t, T_1) = L(t, T_1).$$

Problem (Build the term structure of discount factors)

Given the following strip of FRA rates
FRA(0, 0, 0.5)=4.95%,
FRA(0, 0.5, 1)=5.00%,
FRA(0, 1, 1.5)=5.10%,
FRA(0, 1.5, 2)=5.20%,
build the discount curve up to 2 years.

Example (Computing $P(0, 0.5)$)

Given FRA(0, 0, 0.5) we have immediately the spot discount factor $P(0, 0.5)$

$$P(0, 0.5) = \frac{1}{1 + 0.0495 \times 0.5} = 0.9758478.$$

Then we recursively use the relationship (2).

Example (Computing $P(0,1)$)

The 1 yr discount factor can be computed as follow

$$\begin{aligned} P(0,1) &= \frac{1}{1+F(0,0.5,1)\times 0.5}\times P(0,0.5) \\ &= \frac{1}{1+0.05\times 0.5}\times 0.9758478 \\ &= 0.9520466. \end{aligned}$$

Example (Computing $P(0,1.5)$)

The 1.5 yrs discount factor can be computed as follow

$$\begin{aligned} P(0,1.5) &= \frac{1}{1+F(0,1,1.5)\times 0.5}\times P(0,1) \\ &= \frac{1}{1+0.0510\times 0.5}\times 0.9520466 \\ &= 0.9283731. \end{aligned}$$

Example (Computing $P(0,2)$)

The 2 yrs discount factor can be computed as follow

$$\begin{aligned} P(0,2) &= \frac{1}{1+F(0,1.5,2)\times 0.5} \times P(0,1.5) \\ &= \frac{1}{1+0.0520\times 0.5} \times 0.9283731 \\ &= 0.9048471. \end{aligned}$$

Example (The term structure)

In conclusion, the term structure of discount factors is given by

Time to maturity T	$P(0,T)$
0.5	0.9758478
1	0.9520466
1.5	0.9283731
2	0.9048471

Case Study

Example (Using FRA rates to build the discount curve)

Trade Date: Oct. 12th, 2010 (Tuesday)

Depo, Ois e Fra

	Depo		Ois*			Fra (7)	
Scad.	Den.	Lett.	Den.	Lett.	Scad.	Den.	Lett.
Euro							
1 mese	2,04	2,09	2,06	2,07	3x6	2,22	2,24
2 mesi	2,06	2,11	2,07	2,08	6x9	2,41	2,42
3 mesi	2,10	2,15	2,08	2,09	9x12	2,64	2,66
6 mesi	2,15	2,20	2,13	2,14	6x12	2,55	2,56
12 mesi	2,32	2,37	2,32	2,33	12x18	3,05	3,07

Build the discount curve up to 18 months. The numerical example is available in the Excel file BasicYields.xlsm, sheet: Computing DF from FRA

Example (1. Build the payment schedule)

Value Date: Oct. 14th, 2010 (Thursday)

Start	End	Tenor (months)	Start	End	Adjusted Date	Days
0	3	3	Th, 14 Oct. 2010	Fr, 14 Jan. 2011	Fr, 14 Jan 2011	92
3	6	3	Fr, 14 Jan 2011	Th, 14 April 2011	Th, 14 Apr 2011	90
6	9	3	Th, 14 Ap 2011	Th, 14 July 2011	Th, 14 Jul 2011	91
9	12	3	Th, 14 Jul 2011	Fr, 14 Oct 2011	Fr, 14 Oct 2011	92
12	18	6	Fr, 14 Oct 2011	Sat, 14 Apr 2012	M., 16 Apr 2012	185

Notice that the first expiry is covered by a Deposit rate.

Example (2. Compute Forward Discount Factors)

Forward discount factors are computed according to the formula

$$\frac{1}{1 + FRA\% \times \frac{days}{360}}.$$

Period	Days	FRA	$P(t, T_i, T_{i+1})$
0x3	92	2.1	0.99466
3x6	90	2.22	0.99448
6x9	91	2.41	0.99394
9x12	92	2.64	0.99330
12x18	185	3.05	0.98457

For example, the 9x12 discount factor is

$$\frac{1}{1 + 2.64\% \times \frac{92}{360}} = 0.99330.$$

Example (3. Compute Spot Discount Factors)

Spot discount factors are computed according to the formula

$$P(t, T_i) = P(t, T_{i-1}) \times P(t, T_{i-1}, T_i).$$

Period	Days	FRA	$P(t, T_1, T_2)$	$P(t, T_2)$
0x3	92	2.1	0.99466	0.99466
3x6	90	2.22	0.99448	0.98917
6x9	91	2.41	0.99394	0.98318
9x12	92	2.64	0.99330	0.97659
12x18	185	3.05	0.98457	0.96152

For example, the 0x12 discount factor is

$$0.98318 \times 0.99330 = 0.97659.$$

Application 3: Pricing FRN

Floating Rate Note

- A floater is a debt obligation whose coupon rate is reset at designated dates based on the value of some designated reference rate.
- A FRN is specified by
  1. a number of reset dates $T_0, T_1, \ldots T_{n-1}$,
  2. a number of payment dates $T_1, T_2, \ldots T_n$,
  3. a nominal value N,
  4. a formula for computing the payments at dates $T_1, \ldots T_n$, given the reference rate that resets at dates $T_0, \ldots T_{n-1}$,
     - **At coupon dates:**
       $$c(T_i) = \alpha_{T_{i-1},T_i} \times (L(T_{i-1}, T_i) + \delta) \times N, i = 1, \ldots, n-1,$$
     - **At maturity:**
       $$c(T_n) = \left(1 + \alpha_{T_{n-1},T_n} \times (L(T_{n-1}, T_n) + \delta)\right) \times N$$

  where δ is a fixed interest rate margin (`spread`), and L is the reference rate.
- Notice the natural time lag between reset date T_{i-1} and payment date T_i.

An example of a Floating Rate Note

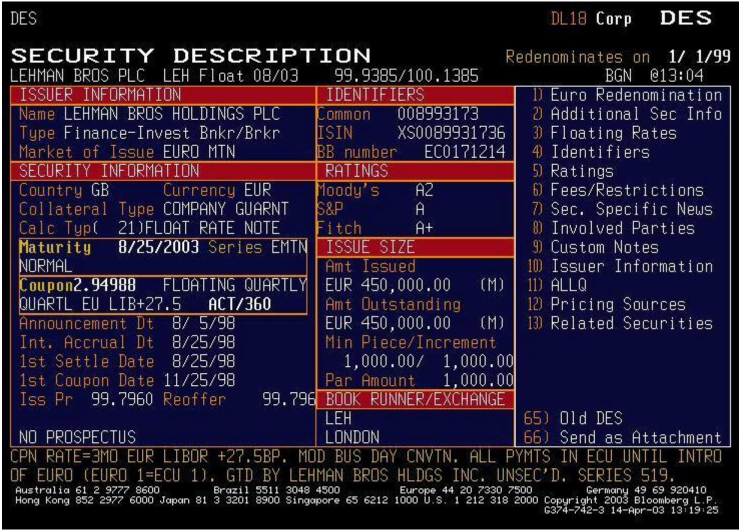


Figure: Floating rate bond

The bond schedule I

- The coupon payment at date T_i is equal to

$$C(T_i) = \left(\text{Reference Rate}(T_{i-1}, T_i) + \frac{27.5}{10000}\right) \times \alpha_{i-1,i} \times N.$$

- The reference rate is the EUR LIBOR with a 3 months tenor. Its value at **reset date** T_{i-1} determines the coupon amount at the payment date T_i.
- Let us suppose that the **Settlement Date** is April 14th, 2003. Then:

  the next coupon is due on Monday, May 26th 2003;

  The coupon starts to accrue on Tuesday Feb 25 2003;

  The reset occurs 2 business days before, i.e. on Friday Feb 21 2003.

The bond schedule II

- On this date, the observed 12 month EUR LIBOR rate was set at 2.67488%.
- Therefore the coupon to be paid on on May 26th, 2003 is equal to

$$(2.67488\% + 0.275\%) \times \frac{90}{360} \times 100 = 2.94988\% \times 0.25 \times 100 = 0.7374.7$$

- Here, the accrual factor $\alpha_{i-1,i}$ is computed according to the ACT/360 convention, i.e. the number of days between two successive coupon dates (adjusted for holidays and weekends) divided by 360.
- Notice that on April 14th, 2003 we do not know the coupon that will be paid on Aug 25th, 2003. The reference rate will be reset on May 2003.

The bond schedule III

Cash Flows of a Floating Rate Note						
Settlement Date	14/04/2003					
Issue Date	Aug. 25th, 1998					
Maturity Date	Aug. 25th, 2003					
Frequency	4					
Current Coupon	2.94988					
First Accrual Date	Aug 25th, 1998					
Day Count Convention	ACT/ACT					
Coupon Dates	Adj. Coupon Date	Days	Coupon	Notional	Tenor	Cash Flow
tu 25 Feb 2003	tu 25 Feb 2003					
su 25 May 2003	m 26 May 2003	90	2.94988	0	0.25	0.73747
m 25 Aug 2003	m 25 Aug 2003	91	**?**	100	0.2527	$100 + ? \times 0.25277 \times 100$

Question.

Given that the 3m EURIBOR rate on the reset date (May 22nd 2003) is equal to 2.35688%, what is the coupon payment on August 25th, 2003?

Question I

- A FRN is paying EURIBOR12m+100bp.
- The coupon is due on October, 20th of each year.
- Determine the coupons that will be paid in October 2000, 2001, 2002, 2003 and 2004.
- Reset days occur four business days before the start of the coupon period. Euribor is rounded to the second digit.
- Payment occurs at the end of the coupon period (reset in advance and pay in arrears).
- Day count convention is ACT/360.
- In the following Table you can read the 12m Euribor rate across different days in October 1999, 2000, 2001, 2002 and 2003.
- **Determine the coupons that have been paid over the years.** (This can be done only ex-post).

Question II

Day	Year				
	1999	2000	2001	2002	2003
22	3.77		3.333	3.168	2.349
21	3.793			3.145	2.362
20	3.772	5.225			2.376
19	3.738	5.209	3.353		
18	3.739	5.189	3.399	3.175	
17		5.201	3.408	3.315	2.372
16		5.194	3.405	3.23	2.338
15	3.755		3.427	3.203	2.308
14	3.714			3.165	2.31

Table: The Table provides the 12m Euribor rate across different days in October 1999, 2000, 2001, 2002 and 2003. Empty cells refer to weekends.

Answer I

Day	Year 1999	2000	2001	2002	2003
22	3.77		**3.333**	3.168	2.349
21	3.793			**3.145**	2.362
20	3.772	**5.225**			**2.376**
19	3.738	5.209	3.353		
18	3.739	5.189	3.399	3.175	
17		5.201	3.408	3.315	2.372
16		5.194	3.405	3.23	2.338
15	3.755		3.427	3.203	2.308
14	3.714			3.165	2.31

Table: 12m Euribor rate in October over different years. Bold are the payment dates adjusted for weekends (Rule: following business day). Yellow cells refer to reset dates.

Answer II

The coupons are

- October 2000:

a. The Euribor rate on Oct. 15th 1999 is 3.755%.

b. The coupon stars to accrue on Oct. 20, 1999.

c. The coupon is paid on Oct. 20, 2000 (i.e. 366 days later).

d. The coupon is equal to $\mathbf{3.5}\% \times \frac{366}{360}$.

- October 2001:

a. The Euribor rate on Oct. 16th 2000 is 5.194%.

b. The coupon stars to accrue on Oct. 20, 2000.

c. The coupon is paid on Oct. 22 (Monday), 2000 (i.e. 367 days later).

d. The coupon is equal to $\mathbf{5.194}\% \times \frac{367}{360}$.

Answer III

- October 2002:

a. The Euribor rate on Oct. 16th 2001 is 3.405%, rounded to 3.41%.

b. The coupon stars to accrue on Oct. 22, 2000 (Monday).

c. The coupon is paid on Oct. 21 (Monday), 2001 (i.e. 364 days later).

d. The coupon is equal to $\mathbf{3.91}\% \times \frac{364}{360}$.

- October 2003:

a. The Euribor rate on Oct. 15th 2002 is 3.203%, rounded to 3.2%.

b. The coupon stars to accrue on Oct. 21, 2002 (Monday).

c. The coupon is paid on Oct. 20, 2003 (i.e. 364 days later).

d. The coupon is equal to $\mathbf{3.2}\% \times \frac{364}{360}$.

- October 2004:

a. The Euribor rate on Oct. 14th 2003 is 2.31%.

b. The coupon stars to accrue on Oct. 20, 2003 (Monday).

c. The coupon is paid on Oct. 20 (wed), 2004 (i.e. 366 days later).

d. The coupon is equal to $\mathbf{2.31}\% \times \frac{366}{360}$.

Pricing a FRN

We have to consider three components

- the current coupon
- the not yet resetted floating coupons
- the face value.

Cash Flows and Fair Value of a FRN I

Table: Cash Flows of the FRN: t is the pricing date, $T_1 > t$ is the next payment date, T_n is the bond expiry, $T_0 < t$ is the last time at which the lastly reset the coupon, c is the current coupon

T_{i-1} (Reset)	T_i (Payment)	$P(t, T_i)$	Cash Flows	Fair Value
t				
T_0	T_1	$P(t, T_1)$	$c \times \alpha_{T_0, T_1}$	$P(t, T_1) \times c \times \alpha_{T_0, T_1}$
T_1	T_2	$P(t, T_2)$	$L(T_1, T_2) \times \alpha_{T_1, T_2}$	$P(t, T_1) - P(t, T_2)$
...	...	...	...	...
T_{i-1}	T_i	$P(t, T_i)$	$L(T_{i-1}, T_i) \times \alpha_{T_{i-1}, T_i}$	$P(t, T_{i-1}) - P(t, T_i)$
...	...	...	...	...
T_{n-1}	T_n	$P(t, T_n)$	$L(T_{n-1}, T_n) \times \alpha_{T_{n-1}, T_n}$	$P(t, T_{n-1}) - P(t, T_n)$
T_{n-1}	T_n	$P(t, T_n)$	1	$1 \times P(t, T_n)$
			Fair Value	...

Cash Flows and Fair Value of a FRN II

- The Fair Value of the FRN is obtained by summing the present value of the three components:

1. The present value of the current coupon

$$P(t, T_1) \times c \times \alpha_{T_0, T_1}.$$

2. The present value of the floating payments

$$\sum_{i=2}^{n} \left(P(t, T_{i-1}) - P(t, T_i)\right) = P(t, T_1) - P(t, T_n)$$

3. The present value of the face value at expiry

$$1 \times P(t, T_n).$$

Cash Flows and Fair Value of a FRN III

- Summing [1]+[2]+[3], we have

$$P(t,T_1)\times c\times\alpha_{T_0,T_1}+P(t,T_1)-P(t,T_n)+P(t,T_n)$$

- Therefore the fair value of the FRN is

$$P(t,T_1)\times c\times\alpha_{T_0,T_1}+P(t,T_1)=P(t,T_1)\times(1+c\times\alpha_{T_0,T_1}).$$

Main Result: **Pricing a FRN is like pricing a zero-coupon bond expiring at the next coupon date (i.e. T_1) and having a face value equal to $1+c\times\alpha_{T_0,T_1}$.**

- This result is valid assuming no spread is paid on the top of the LIBOR rate.

Example (Pricing a FRN)

Let us consider the following term structure of discount factors

Period	Days	FRA	$P(t, T_1, T_2)$	$P(t, T_2)$
0x3	92	2.1	0.99466	0.99466
3x6	90	2.22	0.99448	0.98917
6x9	91	2.41	0.99394	0.98318
9x12	92	2.64	0.99330	0.97659

- We have to determine the market value of a FRN having quarterly coupons and expiring in 12 months, current coupon is 2.1%.
- We have

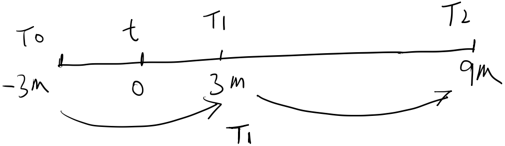
$$0.99466 \times \left(1 + 0.021 \times \frac{92}{360}\right) = 0.99466 \times 1.0054 = 1$$

- Why exactly 1?

Question

Using the same information as before, price a FRN having semi-annual coupons and expiring in 9 months.

$$0.99466\,(1 + 2.1\% \times \alpha_{T_0,T_1}) \qquad \alpha_{T_0,T_1} = \frac{6}{12}$$



Answer

Including the spread

- If the floating coupon due at time T_i includes a spread component

$$\left(L(T_{i-1}, T_i) + \delta\right) \alpha_{T_{i-1}, T_i},$$

we need to modify the pricing formula considering that

1. the current coupon to be paid in T_1 already includes the spread;
2. therefore, the present value of the spread component is like the present value of an annuity starting in T_2 and ending in T_n

$$\delta \sum_{i=2}^{n} P(t, T_i) \alpha_{T_{i-1}, T_i}$$

- Therefore the present value of the floating rate note paying LIBOR+Spread is

$$FRN(t) = P(t, T_1) \times (1 + c \times \alpha_{T_0, T_1}) + \delta \sum_{i=2}^{n} P(t, T_i) \alpha_{T_{i-1}, T_i}.$$

Take Away: Pricing a non-defaultable FRN I

1. **Pricing of non-defaultable FRN at a generic date.** The present value of a default-free floating rate note is

$$P(t, T_1) \times (1 + \alpha_{T_0, T_1} \times c) + \sum_{\mathbf{i=2}}^{n} P(t, T_i) \times \alpha_{i-1,i} \times \delta,$$

where $c = L(T_0, T_1) + \delta$ is the current coupon, i.e. the coupon (inclusive of the spread), determined in T_0 and to be paid in T_1 $T_0 < t < T_1$.

Take Away: Pricing a non-defaultable FRN II

2. **Pricing of a non-defaultable FRN with zero-spread.** If there is no spread component, the previous formula simplifies into

$$P(t, T_1) \times (1 + \alpha_{T_0, T_1} \times c),$$

where $c = L(T_0, T_1)$ is the current coupon, i.e. the coupon determined in T_0 and to be paid in T_1 $T_0 < t < T_1$.

Take Away: Pricing a non-defaultable FRN III

3. **Pricing of non-defaultable FRN at coupon dates.** At the reset date, i.e. $t = T_0$, the present value of a default-free floating rate note simplifies into

$$1 + \sum_{\mathbf{i=1}}^{n} P(t, T_i) \times \alpha_{i-1,i} \times \delta,$$

indeed $P(t, T_1) \times (1 + \alpha_{T_0,T_1} \times c) = P(t = T_0, T_1) \times (1 + \alpha_{T_0,T_1} \times (L(T_0, T_1) + \delta)) = 1 + P(t = T_0, T_1)\, \alpha_{T_0,T_1} \delta.$

Take Away: Pricing a non-defaultable FRN IV

- * No counterparty risk
- * No lag between reset and payment ?
- * Tenor of the reference rate = Tenor of the coupon reset date

4. **Pricing of non-defaultable FRN at coupon dates and zero spread.** In addition, if $\delta = 0$, the FRN at payment dates (i.e. $t = T_0$) is quoted at par

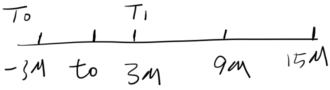
1.

5. The duration of a floating-rate note is the time to wait until the next reset period, at which time the FRN should be at par.

E.g.:

- S-A cash flows
- $\delta = 1\%$
- Expiry 15M
- $C = 2\%$

TTM	P
3M	0.99
6M	0.98
9M	0.97
12M	0.96
15M	0.95



$$0.99\times\left(1+2\%\times\frac{6}{12}\right)+\delta\sum_{2}^{n}P(t,T_i)\times\alpha_{i-1,i}$$

Application 4: Pricing Swaps

Swap I

Table: Cash Flows of the Payer Swap: t is the pricing date, $T_0 \geq t$ is the first reset date, $T_1 > t$ is the first payment date, T_n is the swap expiry, $S(t)$ is the fixed rate (swap rate)

T_{i-1}	T_i	$P(t, T_i)$	Floating Cash Flows	Fixed Cash Flows
t			**0**	**0**
T_0	T_1	$P(t, T_1)$	$L(T_0, T_2) \times \alpha_{T_0, T_1}$	$S(t) \times \alpha_{T_0, T_1}$
T_1	T_2	$P(t, T_2)$	$L(T_1, T_2) \times \alpha_{T_1, T_2}$	$S(t) \times \alpha_{T_1, T_2}$
...	...	...	...	...
T_{i-1}	T_i	$P(t, T_i)$	$L(T_{i-1}, T_i) \times \alpha_{T_{i-1}, T_i}$	$S(t) \times \alpha_{T_{i-1}, T_i}$
...	...	...	...	...
T_{n-1}	T_n	$P(t, T_n)$	$L(T_{n-1}, T_n) \times \alpha_{T_{n-1}, T_n}$	$S(t) \times \alpha_{T_{n-1}, T_n}$
		Fair Value	...	...

Swap II

- The Fair Value of the Payer Swap is obtained by taking the difference of the present value of the two components:

1. The present value of the floating leg:

$$P(t, T_0) - P(t, T_n).$$

2. The present value of the fixed leg:

$$S(t) \sum_{i=1}^{n} P(t, T_i) \times \alpha_{T_{i-1}, T_i}.$$

swap rate

Annuity

- Therefore the value of the payer swap at inception is

$$P(t, T_0) - P(t, T_n) - S(t) \times \sum_{i=1}^{n} P(t, T_i) \times \alpha_{T_{i-1}, T_i}.$$

Swap III

- At inception, the swap rate $S(t)$ is chosen so that the contract has zero value

$$S(t) = \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} P(t, T_i) \times \alpha_{T_{i-1}, T_i}}.$$

pv (FL)

pv (Fixed L)

- $S(t)$ is called forward swap rate.
- If the first reset occurs in t, i.e. $t = T_0$, S(t) is called spot swap rate and is given by

$$S(t) = \frac{1 - P(t, T_n)}{\sum_{i=1}^{n} P(t, T_i) \times \alpha_{T_{i-1}, T_i}}.$$

Example (Computing the swap rate)

Let us consider the following term structure of discount factors

Period	Days	FRA	$P(t, T_1, T_2)$	$P(t, T_2)$
0x3	92	2.1	0.99466	0.99466
3x6	90	2.22	0.99448	0.98917
6x9	91	2.41	0.99394	0.98318
9x12	92	2.64	0.99330	0.97659

- We have to determine the fixed swap rate on the payer swap. First payment resets in 3 months and last payment occurs in 12 months. Payment frequency is quarterly.

Example (Fair Value of the two legs)

- The floating leg is worth

$$0.99466 - 0.97659 = 0.0181.$$

- The fixed leg is worth

$$S \times \left(0.98917 \times \frac{90}{360} + 0.98318 \times \frac{91}{360} + 0.97659 \times \frac{92}{360}\right) = S(t) \times 0.7454$$

- The forward swap rate is

$$S(t) = \frac{0.0181}{0.7454} = 2.42\%.$$

Seasoned Swap

- In a seasoned swap, the payment due at time T_1 has been already resetted in $T_0 < t$, and only the cash flows from T_2 onwards are unknown.
- In this case, the fair value of the IRS is (assuming fixed and floating cash flows have the same frequency and the same day count convention)

$$
\begin{array}{ll}
(\mathbf{a}) & P(t,T_1)\left(L(T_0,T_1)-K\right)\alpha_{T_0,T_1} \\
(\mathbf{b}) & +P(t,T_1)-P(t,T_n) \\
(\mathbf{c}) & -K\sum_{2}^{n} P(t,T_i)\alpha_{T_{i-1},T_i}
\end{array}
$$

where

(a) =present value of the known cash flow,

(b) =present value of the remaining floating payments,

(c) =present value of the remaining fixed payments.

Example (Pricing a seasoned swap)

- We have to price an IRS with semi-annual cash flows, expiring in 9 months. The fixed rate is 3%. The 6m LIBOR that resetted 3 months ago was 3.5%.
- Given the following term structure of discount factors price the seasoned swap

Term (months)	$P(t,T)$
3	0.99466
6	0.98917
9	0.98318
12	0.97659

Example (...(continued))

- The swap has still 2 payment dates: 3 and 9 months.
- **(a):** The present value of the payment due in 3m is

$$0.99466 \times (3.5\% - 3\%) \times 0.5 = 0.00249.$$

- **(b):** The present value of the floating payment due in 9m is

$$0.99466 - 0.98318 = 0.01148.$$

- **(c):** The present value of the fixed payment due in 9m is

$$0.98318 \times 3\% \times 0.5.$$

- Summing up, the present value of the seasoned swap is

$$(a) + (b) - (c) = 0.00249 + 0.01148 - 0.01475 = -0.00078.$$

A Case Study
Pricing an Interest Rate Swap

Example (1. Pricing a Swap)

See Excel file: FI_Swap.xlsm, Sheet: Pricing a Swap

Table: Swap Characteristics

Inception Date	Nov. 5th 2004	
Maturity Date	Nov. 5th 2008	
Trade Date	Feb. 4th 2005 (Friday)	
Value Date	**Feb. 8th 2005**	
Frequency fixed payments	S.A.	
Frequency floating payments	S.A.	
Basis Fixed Leg	30/360	
Basis Floating Leg	ACY/360	
Swap Rate	4.50%	quoted on inception
6m LIBOR rate	4.25%	quoted on reset date

Example (2. Market Data)

Table: Term structure of spot rates (annually compounded) and discount factors on the value date. Bold dates are swap payment dates adjusted for weekends. Payment dates are set with reference to the inception date and NOT the value date.

Payment Dates	Maturity (days)	Spot Rates	DF
05/05/2005	86	4.00%	0.99080
07/11/2005	272	4.20%	0.96981
05/05/2006	451	4.30%	0.94931
06/11/2006	636	4.50%	0.92617
07/05/2007	818	4.80%	0.90026
05/11/2007	1000	5.00%	0.87488
05/05/2008	1182	5.00%	0.85385
05/11/2008	1366	5.00%	0.83311

In the Table, we have for example $0.83311 = \frac{1}{(1+0.05)^{\frac{1366}{365}}}$.

Example (2. Pricing the fixed leg)

Table: Computing the Present Value of the Fixed Leg

Payment Dates	DF	Swap Rate	Accrual Factor	$DF \times S \times AF$
05/05/2005	0.99080	4.5%	0.50000	0.02229
07/11/2005	0.96981	4.5%	0.50556	0.02206
05/05/2006	0.94931	4.5%	0.49444	0.02112
06/11/2006	0.92617	4.5%	0.50278	0.02095
07/05/2007	0.90026	4.5%	0.50278	0.02037
05/11/2007	0.87488	4.5%	0.49444	0.01947
05/05/2008	0.85385	4.5%	0.50000	0.01921
05/11/2008	0.83311	4.5%	0.50000	0.01874
			PV(fixed Leg)	**0.16422**

Example (4. Pricing the Floating Leg)

We price the floating leg using the formula $\frac{P(t,T_1)}{P(T_0,T_1)} - P(t,T_n)$. In particular, we have

- $P(T_0, T_1)$ has been computed using the LIBOR rate at the last reset date (the inception date)

$$P(T_0, T_1) = \frac{1}{1 + 0.0425 \times \frac{181}{360}} = 0.97908$$

- The remaining discount factors come from the previously computed term structure.

	Start Date	End Date	Maturity (days)	Discount factor
$P(t_0, T_1)$	05/11/2004	05/05/2005	181	0.97908
$P(t, T_1)$	08/02/2005	05/05/2005	86	0.99080
$P(t, T_n)$	08/02/2005	05/11/2008	1366	0.83311
			PV(float leg)	**0.17887**

Example (5. Pricing the Payer swap)

The market value of the swap is

$$(0.17887 - 0.16422) \times 1,000,000 = 14,644.$$

Valuation	
PV(fixed)	0.16422
PV(float)	0.17887
Nominal	1,000,000
Swap Value	**14,644**

Example (6. Computing the fixed cash flows)

The Fixed payments are computed according to

$$\text{Swap Rate} \times \frac{30}{360} \times 1,000,000.$$

Start Date	End Date	Accrual Factor	Swap Rate	Fixed Payment
05/11/2004	05/05/2005	0.50000	4.5%	22,500
05/05/2005	07/11/2005	0.50556	4.5%	22,750
07/11/2005	05/05/2006	0.49444	4.5%	22,250
05/05/2006	06/11/2006	0.50278	4.5%	22,625
06/11/2006	07/05/2007	0.50278	4.5%	22,625
07/05/2007	05/11/2007	0.49444	4.5%	22,250
05/11/2007	05/05/2008	0.50000	4.5%	22,500
05/05/2008	05/11/2008	0.50000	4.5%	22,500

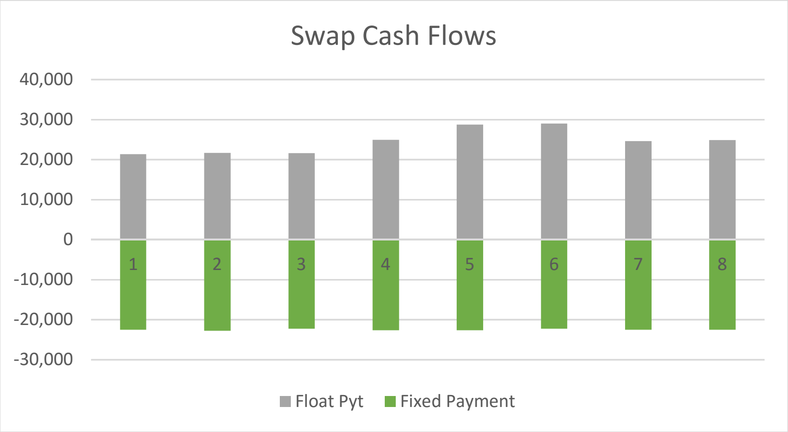
Example (7. Projecting the floating cash flows)

- The first Rate is the LIBOR at the reset date.
- The remaining rates are the simple forward rates computed out of the discount curve.
- The Floating payment is computed according to

$$Rate \times \frac{ACT}{360} \times 1,000,000$$

Start Date	End Dates	Days	Rate	Float Pyt
05/11/2004	05/05/2005	181	**4.2500%**	21,368
05/05/2005	07/11/2005	186	4.1902%	21,649
07/11/2005	05/05/2006	179	4.3424%	21,591
05/05/2006	06/11/2006	185	4.8618%	24,984
06/11/2006	07/05/2007	182	5.6926%	28,779
07/05/2007	05/11/2007	182	5.7390%	29,014
05/11/2007	05/05/2008	182	4.8712%	24,627
05/05/2008	05/11/2008	184	4.8718%	24,901

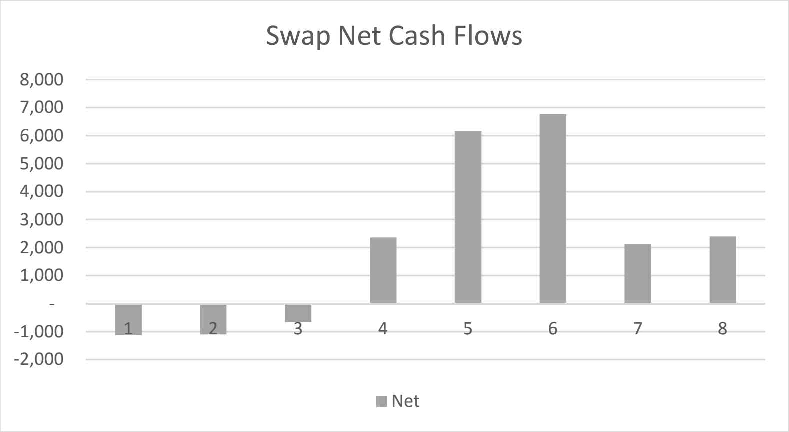
Example


Swap Cash Flows
40,000
30,000
20,000
10,000
0
-10,000
-20,000
-30,000
1
2
3
4
5
6
7
8
Float Pyt
Fixed Payment

Example (8. Net cash flows)

Start Date	End Date	Float Pyt	Fixed Payment	Net	DF	PV
05/11/2004	05/05/2005	21,368	22,500	- 1,132	0.99080	- 1,122
05/05/2005	07/11/2005	21,649	22,750	- 1,101	0.96981	- 1,068
07/11/2005	05/05/2006	21,591	22,250	- 659	0.94931	- 625
05/05/2006	06/11/2006	24,984	22,625	2,359	0.92617	2,185
06/11/2006	07/05/2007	28,779	22,625	6,154	0.90026	5,540
07/05/2007	05/11/2007	29,014	22,250	6,764	0.87488	5,918
05/11/2007	05/05/2008	24,627	22,500	2,127	0.85385	1,816
05/05/2008	05/11/2008	24,901	22,500	2,401	0.83311	2,000
						14,644

Example


Swap Net Cash Flows
8,000
7,000
6,000
5,000
4,000
3,000
2,000
1,000
-
-1,000
-2,000
1
2
3
4
5
6
7
8
Net

Conclusion

- We have seen how to price a flow contract
- We have applied the formula to price several contracts, such as

a Forward Rate Agreements;

b Floating Rate Notes;

c Interest Rate Swaps.

Appendix

Basic Products: Interest Rate Swaps

- IRS consists in an agreement between two counterparts to exchange a series of cash flows on pre-agreed dates in the future.
- The most commonly traded swap (`plain vanilla interest swap`) requires one side to pay a fixed rate and the other to pay a floating rate.

```
   Payer swap:  pays fixed and receive floating,
Receiver swap:  pays floating and receive fixed.
```

- This is fixed versus floating swap with the floating payment based on either three-month, or six-month or 1 year LIBOR rate.
- There are no cash-flows at inception.
- The fixed rate is called the swap rate, while the floating rate is typically a Libor rate.

Example (Plain Vanilla US Interest Swap)

- Floating Payments
  - are tied to three months LIBOR;
  - are made at three months intervals;
  - determined three months and two days before each payment date;
  - day count is actual/360.
- Fixed Payments
  - are made at six months intervals.
  - day count is 30/360.

Example (US Swap with 2 years tenor)

Table: Payment dates and cash flows for a US swap with 2 years tenor

	Fixed	Floating	Net Cash Flows
0			
0.25		x	$-N \times L(0; 0.25) \times 0.25$
0.5	x	x	$N \times K \times 0.5 - N \times L(0.25; 0.5) \times 0.25$
0.75		x	$-N \times L(0.5; 0.75) \times 0.25$
1	x	x	$N \times K \times 0.5 - N \times L(0.75; 1) \times 0.25$

Example (Plain Vanilla Euro Interest Swap)

- Floating Payments
  - are tied to six months LIBOR;
  - are made at six months intervals;
  - determined six months and two days before each payment date;
  - day count is actual/360.
- Fixed Payments
  - are made at annual intervals.
  - day count is 30/360.

Example (Euro Swap with 2 years tenor)

Table: Payment dates and cash flows for a US swap with 2 years tenor

	Fixed	Floating	Net Cash Flows
0			
0.5		x	$-N \times L(0; 0.5) \times 0.5$
1	x	x	$N \times K \times 1 - N \times L(0.5; 1) \times 0.5$
1.5		x	$-N \times L(1; 1.5) \times 0.5$
2	x	x	$N \times K \times 1 - N \times L(1.5; 2) \times 0.5$

Contract basic features I

- N : nominal
- Business day convention
- Dates:
  - t : settlement date (or effective date)
  - $T_0, \dots, T_{n-1}$: reset dates
  - $T_1, \dots, T_n$: payment dates
  - T_n : maturity (or termination date)
- Swap Tenor: the time distance (in years) between first reset and last payment date, i.e. $T_n - T_0$.
- Spot Starting Swap: if $t = T_0$, i.e. the first reset occurs on the Value Date.
- Forward Starting Swap: if $t < T_0$, i.e. the first reset is forward respect to the Value Date.

Contract basic features II

- Fixed leg (sequence of floating payments):
  - $K = S(t, T_0, T_n)$: swap rate (fixed rate)
  - frequency
  - day count α^{FI}
- Floating leg (sequence of fixed payments):
  - floating rate (e.g., Libor)
  - frequency
  - day count α^{FL}
- Day count conventions and payment frequencies can be different on the two legs.

Time profile and cash flows of a payer IRS

Payer IRS Cash Flows		
t		Trading date
T_0		1^{st} reset date
T_1	$\searrow$	$N\left(\alpha^{FL}_{T_0, T_1}\widetilde{L}(T_0, T_1) - \alpha^{FI}_{T_0, T_1}K\right)$
...	$\searrow$	...
T_{i-1}		
T_i	$\searrow$	$N\left(\alpha^{FL}_{T_{i-1} T_i}\widetilde{L}(T_{i-1}, T_i) - \alpha^{FI}_{T_{i-1}, T_i}K\right)$
T_{i+1}	$\searrow$	
...	$\searrow$	...
T_{n-1}	$\searrow$	...
T_n		$N\left(\alpha^{FL}_{T_{n-1} T_n}\widetilde{L}(T_{n-1}, T_n) - \alpha^{FI}_{T_{n-1} T_n}K\right)$

- Notice that in a spot starting swap $(t = T_0)$ the first payment is known at inception, since the first floating rate resets at T_0, while all future cash flows remains random.

Time profile and cash flows of a receiver IRS

Receiver IRS Cash Flows		
t		Trading date
T_0		1^{st} reset date
T_1	$\searrow$	$N\left(\alpha^{FI}_{T_{0,}T_1}K - \alpha^{FL}_{T_{0,}T_1}L\left(T_0, T_1\right)\right)$
...	$\searrow$	...
T_{i-1}		
T_i	$\searrow$	$N\left(\alpha^{FI}_{T_{i-1}T_i}K - \alpha^{FL}_{T_{i-1,}T_i}L\left(T_{i-1}, T_i\right)\right)$
T_{i+1}	$\searrow$	
...	$\searrow$	...
T_n		$N\left(\alpha^{FI}_{T_{n-1}T_n}K - \alpha^{FL}_{T_{n-1}T_n}L\left(T_{n-1}, T_n\right)\right)$

Table: Payment dates on different swaps

	Tenor 2yr Spot Starting			Tenor 1yr Fwd Starting in 1y			Tenor 1yr Fwd Starting in 1y	
Frequency	Fixed Q	Floating SA		Fixed Q	Floating SA		Fixed SA	Floating SA
0			0			0		
0.25	x		0.25			0.25		
0.5	x	x	0.5			0.5		
0.75	x		0.75			0.75		
1	x	x	1			1		
1.25	x		1.25	x		1.25		
1.5	x	x	1.5	x	x	1.5	x	x
1.75	x		1.75	x		1.75		
2	x	x	2	x	x	2	x	x

Example (1. Cash flows in a four-year vanilla swap)

- Fixed payments are based on a 4.6% semi-annual rate.
- Floating payments are based on 6-month Libor.
- The initial Libor rate is known to be 2.8% at the outset, so the swap's first payment is certain.
- Subsequent Libor rates are not known at the outset. Note that the final Libor rate at 4.0 years is not used to calculate any of the swap's cash flows.
- The last column indicates cash flows to the receive-fixed party.
- Cash flows to the receiver-floating party are the negatives of these.
- All cash flows are in millions of dollars. Note also how all USD 100MM principal payments net to zero.

Example (2. (continued))

Time (yrs)	6M Libor	Fixed Leg	Float. Leg	Net
0	2.8	-100	-100	0
0.5	3.4	2.3	1.4	0.9
1	4.4	2.3	1.7	0.6
1.5	4.2	2.3	2.2	0.1
2	5	2.3	2.1	0.2
2.5	5.6	2.3	2.5	-0.2
3	5.2	2.3	2.8	-0.5
3.5	4.4	2.3	2.6	-0.3
4	3.8	102.3	102.2	0.1

Table: **First column**: relevant dates; **Second column**: 6M LIBOR rates at reset dates; **Third column**: Fixed leg payment, e.g. in the cyan cell $4.6\% \times 0.5 \times 100 = 2.3$; **Fourth column**: Floating leg payment, e.g. in the gree cell $4.4\% \times 0.5 \times 100 = 2.2$; **Fifth column**: Net cashflows (receiver swap).

Uses of IRS

Hedging with swaps I

- IRS allow to optimize the financial condition of debt (when firms have objective and conditions on fixed/floating rate debt).

a. Hedging of fixed-income portfolios against any change in the yield curve by exploiting the DV01 of the swap.
Example. We have a risky position RP with interest rate risk measured by a $DV01_{RP}$. Let us consider a IRS having an interest rate exposure as measured by $DV01_{IRS}$. We can build a portfolio with low interest rate exposure by entering into n IRS so that

$$DV01_{RP} + nDV01_{IRS} = 0,$$

i.e.

$$n = -\frac{DV01_{RP}}{DV01_{IRS}}.$$

Hedging with swaps II

b. They allow to convert the financial conditions of assets and liabilities

- In a plain vanilla IRS there is no exchange of principal.
- However, if we assume that the nominal is both received and paid at the swap maturity, then a swap can be replicated by a portfolio of a floating-rate and a fixed-rate bond.

$$\text{PAYER IRS} = \text{FLOATING-RATE BOND - FIXED-RATE BOND}$$

$$\text{RECEIVER IRS} = \text{FIXED-RATE BOND - FLOATING-RATE BOND}$$

Hedging with swaps III

- For example, we can transform a FRN in a Fixed-rate bond by entering a receiver swap (selling a payer swap):

$$\text{FIXED-RATE BOND} = \text{FLOATING-RATE BOND} + \text{RECEIVER IRS}$$

- Viceversa, we can transform a Fixed Rate Bond into a Floating Rate one:

$$\text{FLOATING-RATE BOND} = \text{FIXED-RATE BOND} + \text{PAYER IRS}$$

Swap rates and market quotations

- Actually the market quotes the spot starting swap rate $S(t, t, T_n)$ for different maturities T_n :

	Euro		GBP Sterling		SwFr		US		Yen	
Tenor T_n	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
1 yr	1.95	2	0.99	1.02	0.45	0.51	0.36	0.39	0.32	0.38
2 yr	2.34	2.39	1.62	1.66	0.86	0.94	0.76	0.79	0.34	0.4
3 yr	2.63	2.68	2.06	2.1	1.18	1.26	1.25	1.28	0.38	0.44
4 yr	2.85	2.9	2.43	2.48	1.45	1.53	1.72	1.75	0.46	0.52
5 yr	3.02	3.07	2.74	2.79	1.69	1.77	2.13	2.16	0.55	0.61
6 yr	3.16	3.21	2.98	3.03	1.89	1.97	2.48	2.51	0.67	0.73
7 yr	3.28	3.33	3.18	3.23	2.05	2.13	2.76	2.79	0.81	0.87
8 yr	3.38	3.43	3.35	3.4	2.17	2.25	2.99	3.02	0.95	1.01
9 yr	3.46	3.51	3.48	3.53	2.27	2.35	3.18	3.21	1.08	1.14
10 yr	3.54	3.59	3.59	3.64	2.36	2.44	3.34	3.37	1.21	1.27
12 yr	3.68	3.73	3.74	3.81	2.49	2.59	3.58	3.61	1.42	1.5
15 yr	3.84	3.89	3.88	3.97	2.61	2.71	3.82	3.85	1.67	1.75
20 yr	3.92	3.97	3.95	4.08	2.67	2.77	4.01	4.04	1.93	2.01
25 yr	3.87	3.92	3.96	4.09	2.67	2.77	4.1	4.13	2.03	2.11
30 yr	3.78	3.83	3.95	4.08	2.65	2.75	4.15	4.18	2.08	2.16

Table: Bid and ask rates as of close of London business (May 2nd, 2011). US $ is quoted annual money actual/360 basis against 3 month Libor. £ and Yen quoted on a semi-annual actual/365 basis against 6 month Libor. Euro/Swiss Franc quoted on annual bond 30/360 basis against 6 month Euribor/Libor with exception of the 1 year rate which is quoted against 3 month Euribor/Libor. Source: ICAP plc. Historical quotes downloadable at `http://markets.ft.com/RESEARCH/Markets/DataArchive`

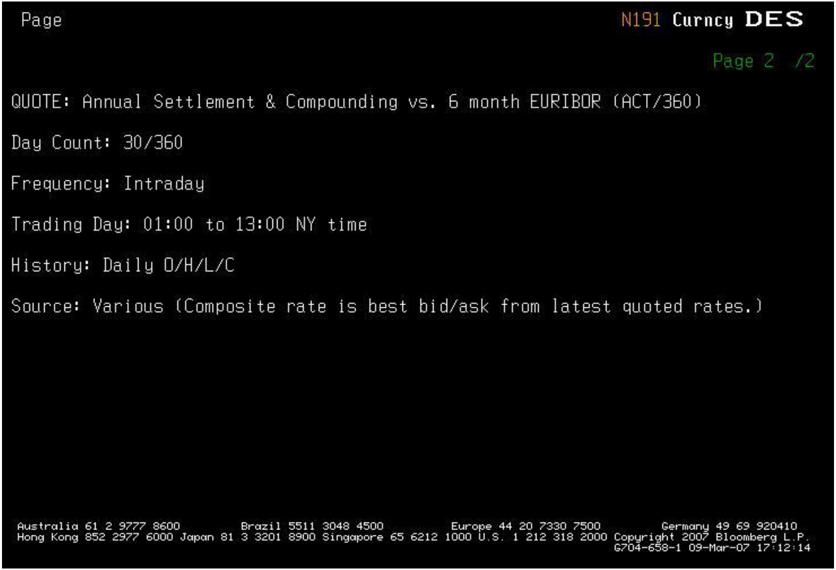
Swap rates and market conventions

- Different payment frequencies, compounding frequencies and day count conventions are applicable to each currency-specific interest rate type.

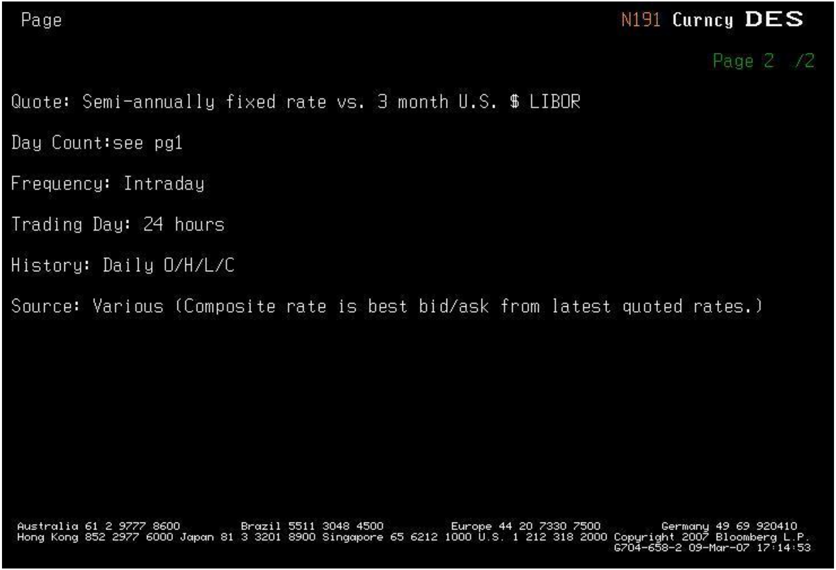
Currency	EURO	JPY	USD	GBP	CHF
Index	EURIBOR or LIBOR	LIBOR or TIBOR	LIBOR	LIBOR	LIBOR
FIXED LEG					
Payment freq.	A	S/A	S/A	A for 1yr then S/A	A
Day Count Convention	$\frac{30}{360}$	$\frac{ACT}{365}$	$\frac{30}{360}$	$\frac{ACT}{365}$	$\frac{30}{360}$
FLOATING LEG					
Payment freq.	3m for 1yr then 6m	6m	3m	6m	3m for 1yr then 6m
Day Count Convention	$\frac{ACT}{360}$	$\frac{ACT}{360}$	$\frac{ACT}{360}$	$\frac{ACT}{365}$	$\frac{ACT}{360}$
Business Days	Target	Tokyo	New York	London	Zurich
Roll Day			modified following		

Table: Quotation Basis for Interest Rate Swaps

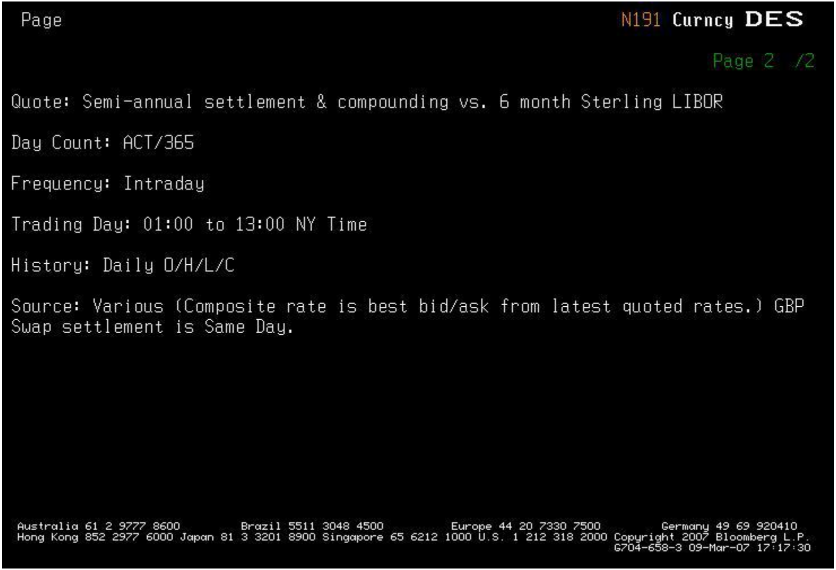
Euro Swap rates: market conventions



USD Swap rates: market conventions



GBP Swap rates: market conventions


Page N191 Curncy DES

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Quote: Semi-annual settlement & compounding vs. 6 month Sterling LIBOR

Day Count: ACT/365

Frequency: Intraday

Trading Day: 01:00 to 13:00 NY Time

History: Daily O/H/L/C

Source: Various (Composite rate is best bid/ask from latest quoted rates.) GBP Swap settlement is Same Day.

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410
Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2007 Bloomberg L.P.
G704-658-3 09-Mar-07 17:17:30

Yen Swap rates: market conventions

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Quote: Semi-annual fixed rate vs. 6 month Yen LIBOR (ACT/365)

TIBOR objects are quoted as: Semi-annual fixed rate vs. 6 month TIBOR (ACT/360)

Day Count: ACT/365

Frequency: Intraday

Trading Day: 24 hours

History: Daily O/H/L/C

Amount: Standard amount for quoted rates is 5 billion yen

Source: Various (Composite rate is best bid/ask from latest quoted rates.)

Pricing away from inception

How to value an IRS after initiation I

- The value of a swap though customarily is set to zero at inception needs not have zero value after inception.
- Let t be the pricing date and T_0 the last reset date ($T_0 < t$).
- The fixed leg has value

$$K \times \sum_{i=1}^{n} \alpha^{FI}_{T_{i-1},T_i} P(t, T_i).$$

- The floating leg has two components
  - the cash flow due in T_1 that resetted in T_0, so it has value

$$P(t, T_1) \times L(T_0, T_1)\, \alpha^{FL}_{T_0,T_1};$$

  - the forward starting floating leg with first reset in T_1 and last payment in T_n has value

$$P(t, T_1) - P(t, T_n).$$

  - the floating leg is worth

$$P(t, T_1)\left(1 + L(T_0, T_1)\, \alpha^{FL}_{T_0,T_1}\right) - P(t, T_n).$$

Value of a Payer Swap away from inception

A payer swap contract having a fixed rate K and that has been set before the current date t has value:

$$P(t, T_1)\left(1 + L\left(T_0, T_1\right)\alpha^{FL}_{T_0, T_1}\right) - P(t, T_n) - K \times \sum_{i=1}^{n} \alpha^{FI}_{T_{i-1}, T_i} P\left(t, T_i\right).$$

i.e. the swap can be replicated using

- a long zcb expiring in T_1 having face value $1 + L\left(T_0, T_1\right)\alpha^{FL}_{T_0, T_1}$;
- shorting a fixed-rate bond having fixed rate K.

Remark If $t = T_0$, the value of the floating leg simplifies to 1, and we get the known result at inception

$$1 - P(t, T_n) - K \times \sum_{i=1}^{n} \alpha^{FI}_{T_{i-1}, T_i} P\left(t, T_i\right) = 0.$$

Hedging Interest Rate Risk
The Dollar Value of 1 Basis Point (DV01)

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SMM269 Fixed Income

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Outline

1. Interest Rate Risk
2. Hedging using DV01
3. Portfolio Hedging using DV01

Reading List

Required Readings

- Any Standard Fixed Income Book.

Accompanying Excel files

- FI_BasicYields.xls
- FI_InterestRateRisk.xls

Interest Rate Risk

- The interest rate risk of a security may be measured by how much its price changes as interest rate changes.
- Measuring the exposure to interest rate risk can be interesting at least for two reasons:
  1. `hedging`: matching the exposure to interest rate risk of the assets with the interest rate risk of the liabilities;
  2. `exploiting views`: given a view avout future changes in interest rates to determine which securities (or combination of securities) will perform best if their view does, in fact, obtain.

DV01: Dollar Value of 1 basis points

- The DV01 (or BPV) of the bond refers to the loss related to 1 basis point shift in the risk factor.
- The most common assumption is to take as risk factor the spot rate relative to maturity T_i.
- So we can define the DV01 relative to the i-th spot rate as

$$DV01_i = -\frac{\partial B}{\partial Y_i} dY_i,$$

  where the negative sign gives a positive value to the DV01.

- In practice, we never compute partial derivatives but we approximate them via finite differences, and we set the shock dY_i equal to 1 b.p., so that

$$DV01_i = -\frac{\Delta B}{\Delta Y_i} \times \frac{1}{10,000},$$

$$B = e^{-y\tau}$$

$$\Delta B \simeq \frac{\partial B}{\partial Y_i} \Delta Y_i \qquad \frac{\partial B}{\partial y} \approx -\tau e^{-y\tau} \qquad DV01 = \tau e^{-y\tau} \times 0.01\%$$

DV01 (or BPV) : Numerical approximation

- A centered difference approximation to the DV01 calculation is done via the following incremental ratio

$$DV01 = -\frac{B\left(y^{*}+\Delta y\right)-B\left(y^{*}-\Delta y\right)}{2\Delta y} \times \frac{0.01}{100}.$$

Example (Numerical Computation of the DV01)

- Let us consider a Bond with 3 years to maturity, 3% coupon, semi-annual coupons and yield to maturity 4%. We can build the following Table. The DV01 is $-\frac{0.9728-0.9733}{2\times 0.01}$.

	YTM	4.01%	4.00%	3.99%
Payment Dates (years)	Cash Flows	PV(CF)	PV(CF)	PV(CF)
0.5	0.015	0.0147	0.0147	0.0147
1	0.015	0.0144	0.0144	0.0144
1.5	0.015	0.0141	0.0141	0.0141
2	0.015	0.0139	0.0139	0.0139
2.5	0.015	0.0136	0.0136	0.0136
3	1.015	0.9021	0.9023	0.9026
		$B(y+dy)$	$B(y)$	$B(y-dy)$
Gross Price		**0.9728**	**0.9731**	**0.9733**

Example

- The centered approximation to the DV01 computation is

$$DV01 = -\frac{0.9728 - 0.9733}{2 \times 0.001} \times \frac{1}{10000} = 0.0003$$

- This is the bond loss due 1 bp change in the term structure
- This means that if the term structure moves up by 20 bps, the bond price will change by the amount

$$-0.0003 \times 20$$

- If the term structure moves down by 15 bps, the bond price will change by the amount

$$-0.0003 \times (-15).$$

Hedging using DV01 I

- We can hedge a risky position using DV01.
- We proceed as follows:
  1. Suppose we have a risky position RP with interest rate risk measured by a $DV01_{RP}$.
  2. Let us consider an interest rate derivative, e.g. an IRS, having an interest rate exposure as measured by $DV01_{IRS}$.
  3. We can build a portfolio with low interest rate exposure by entering into n IRS so that the exposure of the entire portfolio is 0

$$DV01_{RP} + nDV01_{IRS} = 0,$$

i.e.

$$n = -\frac{DV01_{RP}}{DV01_{IRS}}.$$

Hedging using DV01 II

Example (Hedging with DV01)

A bond, with notional 1ml USD, has a DV01 equal to 300 USD. An payer IRS, of notional 10ml, has a DV01 of -200. In order to hedge the interest risk of the bond we need to buy

$$n = -\frac{300}{-200} = 1.5,$$

i.e. 15 ml of notional (=1.5x10ml).

Hedging using DV01 III

- This works if the risk factors RF affecting the values of the two risky positions are perfectly correlated, indeed the profit and loss of the hedged portfolio is

$$P\&L = \Delta RP(t) + n\Delta IRS(t) = -DV01_{RP}\Delta RF_{RP} - nDV01_{IRS}\Delta RF_{IRS}$$

and if $\Delta RF_{RP} = \Delta RF_{IRS} = \Delta RF$, we have

$$P\&L = -(DV01_{RP} - nDV01_{IRS})\Delta RF$$

and with the above choice of n we are perfectly hedged.

- In general, the assumption of perfect correlation is not fully true and we are left with some *basis risk*: market risk is replaced by basis risk.
- A possible procedure to cope with it is to find a minimum variance hedge, i.e. to find n such that the variance of the Profit and Loss is minimized. This procedure is described in the Appendix.
- Other approaches are described in the following pages.

Measuring Interest Rate Risk I

- The challenge in measuring ΔB comes from defining realistic interest rate factors.
- Indeed, the bond price change (and the price of any interest rate derivative) is exposed to a large number of interest shocks and on their joint movements, i.e. how do we take into account a joint movement in Y_i and Y_j?.
- We can define reduced form risk measures assuming particular shifts in the risk factors.

Measuring Interest Rate Risk II

- The DV01 calculation is straightforward assuming that the interest rate changes are
  1. perfectly correlated (there is just one risk factor);
  2. have same sensitivity to this factor, i.e. they move in a `parallel way`, i.e.
- We proceed as follows
  1. given an initial $Y(t, t_1)$,..., $Y(t, t_n)$, and
  2. a shifted spot rate curve $Y(t, t_1) + dY_1$,..., $Y(t, t_n) + dY_n$,
  3. we can reprice the security and compute the price change:

$$\Delta B = \sum_{i=1}^{n} \left(\frac{c_i}{(1 + Y(t, t_i))^{t_i - t}} - \frac{c_i}{(1 + Y(t, t_i) + dY_i)^{t_i - t}} \right),$$

  where $c_i = c/m$, $i = 1, ..., n-1$, and $c_n = c/m + N$

4. We can assume as first approximation a parallel change across all maturities t_i, i.e. a unique factor is moving all spot/forward/swapt rates

$$dY_i = \sigma \times df, \forall i.$$

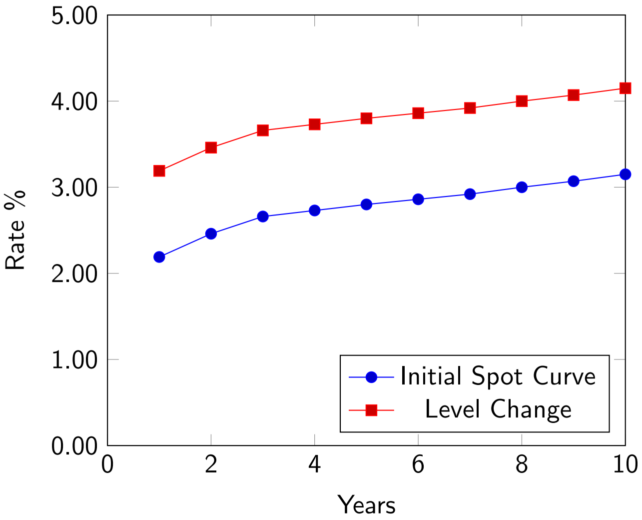


Figure: Parallel shift in the level of the term structure

See Excel File: **FI_InterestRateRisk.xlsm**
Sheet: **Method 1 Parallel Spot Curve**

Example (DV01 via parallel shifts in the spot curve)

Term	Swap	DF	Spot Rates	Shift	Shifted Rates	Shifted DF	CF	PV CF	PV CF (Shift)
		1							
1	-0.296%	1.0030	-0.296%	0.01%	-0.286%	1.0029	5	5.01	5.01
2	-0.184%	1.0037	-0.184%	0.01%	-0.174%	1.0035	5	5.02	5.02
3	-0.095%	1.0029	-0.095%	0.01%	-0.085%	1.0026	5	5.01	5.01
4	0.027%	0.9989	0.027%	0.01%	0.037%	0.9985	5	4.99	4.99
5	0.160%	0.9920	0.161%	0.01%	0.171%	0.9915	5	4.96	4.96
6	0.295%	0.9824	0.297%	0.01%	0.307%	0.9818	5	4.91	4.91
7	0.425%	0.9704	0.429%	0.01%	0.439%	0.9698	5	4.85	4.85
8	0.548%	0.9567	0.555%	0.01%	0.565%	0.9559	5	4.78	4.78
9	0.662%	0.9414	0.673%	0.01%	0.683%	0.9406	5	4.71	4.70
10	0.766%	0.9251	0.781%	0.01%	0.791%	0.9242	105	97.14	97.04
							Bond Price	**141.39**	**141.28**

Table: **Computing the Bond DV01 via shift in the spot rates**. 1. Given the current discount curve price the bond (141.39). 2. Given the current discount curve, compute the spot curve (here the ann. compounded one), shift it, recompute the discount curve and reprice the bond (141.28). 3. The bond DV01 is 0.1181.

Measuring Interest Rate Risk

- Alternative assumptions are:
  1. to assume a parallel shift in forward rates

$$dF(t, T_{i-1}, T_i) = \sigma \times df$$

     and translate changes in forward rates into changes of the spot rates;
  2. to assume a parallel shift in market quotes (e.g. swap rates)

$$dS(t, T_i) = \sigma \times df$$

     and translate these changes in changes of the spot rates;
  3. Yield based DV01, where we assume that the interest rate factor is the yield to maturity;
  4. Assuming different shifts at different term structure pillars (this assumption is supported by the observation that short-term rates are more volatile than long-term rates):

$$dY(t, T_i) = \sigma_i \times df.$$

See Excel File: **FI_InterestRateRisk.xlsm**
Sheet: **Method 2 Parallel Fwd Curve**

Example (DV01 via parallel shift in the forward curve)

Tenor	Swap	DF	Fwd Rates	Shift	Shifted	DF Shift	CF	PV CF	PV CF (Shift)
		1				1			
1	-0.296%	1.0030	-0.296%	0.01%	-0.286%	1.0029	5	5.01	5.01
2	-0.184%	1.0037	-0.072%	0.01%	-0.062%	1.0035	5	5.02	5.02
3	-0.095%	1.0029	0.083%	0.01%	0.093%	1.0026	5	5.01	5.01
4	0.027%	0.9989	0.395%	0.01%	0.405%	0.9985	5	4.99	4.99
5	0.160%	0.9920	0.697%	0.01%	0.707%	0.9915	5	4.96	4.96
6	0.295%	0.9824	0.982%	0.01%	0.992%	0.9818	5	4.91	4.91
7	0.425%	0.9704	1.226%	0.01%	1.236%	0.9698	5	4.85	4.85
8	0.548%	0.9567	1.442%	0.01%	1.452%	0.9559	5	4.78	4.78
9	0.662%	0.9414	1.620%	0.01%	1.630%	0.9406	5	4.71	4.70
10	0.766%	0.9251	1.761%	0.01%	1.771%	0.9242	105	97.14	97.04
							Bond Price	**141.39**	**141.28**

Table: **Computing the Bond DV01 via shift in the forward rates**. 1. Given the current discount curve price the bond (141.39). 2. Given the current discount crurve, compute the forward curve, shift it, recompute the discount curve and reprice the bond (141.28). 3. The bond DV01 is 0.1181=-(141.28-141.39).

Example (Hedging using DV01)

We hedge the exposure using a 10 year swap.

- We set the fixed rate of the swap using the initial discount curve

$$S = \frac{0.07489}{9.77641} = 0.7660\%$$

- The initial value of the swap is 0.
- Using the shifted curve, we reprice the swap.
- The new value of the swap is 0.00096.
- The DV01 of the swap is -0.000957907.
- The Hedge Ratio is $-0.1181/-0.000957907 = 123.34$, i.e. we need a notional of 123 USD to hedge the bond exposure.

Hedging with 10 years swap		
	Initial Curve	Shifted Curve
AF	1	1
Annuity	9.77641	9.77113
Floating Leg	0.07489	0.07580
Swap Rate	0.00766	0.00776
FV	0.00000	0.00096
DV01	-0.00096	
Hedge Ratio	**123.3388**	

See Excel File: **FI_InterestRateRisk.xlsm**
Sheet: **Method 3 Parallel Swap Curve**

Example (DV01 via parallel shift in the swap curve)

Term	Swap	DF	Shift	Shifted Swap	DF Shift	CF	PV CF	PV CF (Shift)
		1			1			
1	-0.296%	1.0030	0.01%	-0.286%	1.0029	5	5.01	5.01
2	-0.184%	1.0037	0.01%	-0.174%	1.0035	5	5.02	5.02
3	-0.095%	1.0029	0.01%	-0.085%	1.0026	5	5.01	5.01
4	0.027%	0.9989	0.01%	0.037%	0.9985	5	4.99	4.99
5	0.160%	0.9920	0.01%	0.170%	0.9915	5	4.96	4.96
6	0.295%	0.9824	0.01%	0.305%	0.9818	5	4.91	4.91
7	0.425%	0.9704	0.01%	0.435%	0.9698	5	4.85	4.85
8	0.548%	0.9567	0.01%	0.558%	0.9559	5	4.78	4.78
9	0.662%	0.9414	0.01%	0.672%	0.9405	5	4.71	4.70
10	0.766%	0.9251	0.01%	0.776%	0.9242	105	97.14	97.04
						Bond Price	**141.39**	**141.27**

Table: **Computing the Bond DV01 via shift in the swap rates**. 1. Given the current swap curve, bootstrap discount factors and price the bond (141.39). 2. Given the current swap curve, shift it and recompute the discount curve via bootstrap and reprice the bond (141.27). 3. The bond DV01 is 0.1203=-(141.27-141.39).

Example (DV01 via shift in the yield to maturity)

Term	Swap	DF	CF	PV CF	YTM	PV CF	Shift	Shifted YTM	PV CF (Shift)
		1							
1	-0.296%	1.0030	5	5.01	0.70%	4.97	0.01%	0.71%	4.96
2	-0.184%	1.0037	5	5.02	0.700%	4.93	0.01%	0.71%	4.93
3	-0.095%	1.0029	5	5.01	0.700%	4.90	0.01%	0.71%	4.90
4	0.027%	0.9989	5	4.99	0.700%	4.86	0.01%	0.71%	4.86
5	0.160%	0.9920	5	4.96	0.700%	4.83	0.01%	0.71%	4.83
6	0.295%	0.9824	5	4.91	0.700%	4.80	0.01%	0.71%	4.79
7	0.425%	0.9704	5	4.85	0.700%	4.76	0.01%	0.71%	4.76
8	0.548%	0.9567	5	4.78	0.700%	4.73	0.01%	0.71%	4.72
9	0.662%	0.9414	5	4.71	0.700%	4.70	0.01%	0.71%	4.69
10	0.766%	0.9251	105	97.14	0.700%	97.93	0.01%	0.71%	97.83
			Bond Price	141.39		**141.39**			**141.27**

Table: **Computing the Bond DV01 via shift in the YTM**. 1. Given the current swap curve, bootstrap discount factors and price the bond (141.39). 2. Compute the bond YTM (0.70%), shift it to (0.71%) and reprice the bond (141.27). 3. The bond DV01 is 0.12=-(141.27-141.39). In this example, we can also compute the bond duration (=8.45), the modified duration (8.39=8.45/(1+0.70%)), and then the DV01 via the approximation $-8.39 \times 141.39 \times 0.01\% = 0.12$. For details see the Excel file FI_InterestRateRisk.xls

Change in the slope

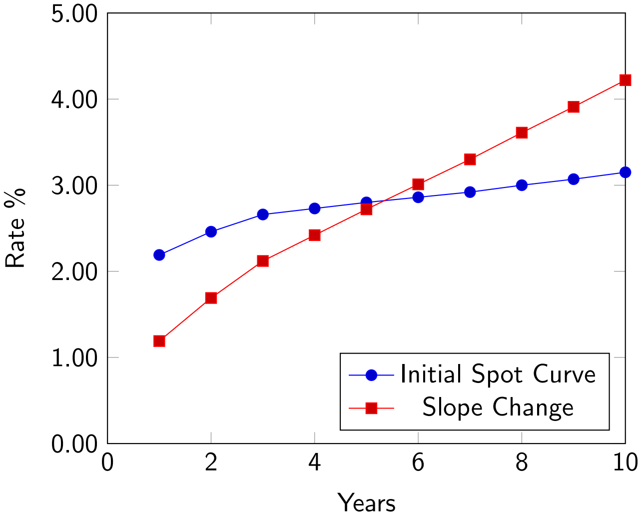


Figure: Change in the slope of the term structure

Building non parallel shift: A slope Change

- A slope change can be captured by a downward movement in the short term and an upward movement in the long term (or viceversa)

$$dY(t, T_i) = \sigma_i \times df,$$

where $\sigma_i > 0$ for some maturities and $\sigma_i < 0$ for other maturities.

Term	Initial	Shift	New
T_1	$Y(t, T_1)$	$\Delta_1 > 0(or < 0)$	$Y(t, T_1) + \Delta_1$
...	...		...
T_i	$Y(t, T_1)$	$\Delta_i = 0$	$Y(t, T_i)$
...	...		...
T_n	$Y(t, T_n)$	$\Delta_n < 0(or > 0)$	$Y(t, T_n) + \Delta_n$

If we set

$$\beta = \frac{\Delta_n - \Delta_1}{T_n - T_1}, \quad \alpha = \frac{T_n \cdot \Delta_1 - T_1 \cdot \Delta_n}{T_n - T_1},$$

then we can model a change in slope by setting

$$\Delta_i = \alpha + \beta \cdot (T_i - t)$$

Example (Building a slope shift)

- We use $T_1 = 1$ as shortest maturity and $T_n = 10$ as the longest one.
- We set $\Delta_1 = 0.20\%$ (20 b.p) and $\Delta_{10} = -0.20\%$ (20 b.p)
- Then we have $\beta = \frac{0.20\% - (-0.20\%)}{10-1} = -0.044\%$, and $\alpha = \frac{10 \cdot 0.20\% - 1 \cdot (-0.20\%)}{10-1} = 0.244\%$ so that the change in the spot rate for a generic maturity is given by

$$\Delta_i = 0.244\% - 0.044\% \cdot (T_i - 0).$$

Tenor	Swap	DF	Fwd Rates	Slope Shift	Shifted	DF Shift	CF	PV CF	PV CF (Shift)
		1				1			
1	-0.296%	1.0030	-0.296%	0.20%	-0.096%	1.0010	5	5.01	5.00
2	-0.184%	1.0037	-0.072%	0.16%	0.083%	1.0001	5	5.02	5.00
3	-0.095%	1.0029	0.083%	0.11%	0.194%	0.9982	5	5.01	4.99
4	0.027%	0.9989	0.395%	0.07%	0.461%	0.9936	5	4.99	4.97
5	0.160%	0.9920	0.697%	0.02%	0.720%	0.9865	5	4.96	4.93
6	0.295%	0.9824	0.982%	-0.02%	0.960%	0.9771	5	4.91	4.89
7	0.425%	0.9704	1.226%	-0.07%	1.160%	0.9659	5	4.85	4.83
8	0.548%	0.9567	1.442%	-0.11%	1.331%	0.9532	5	4.78	4.77
9	0.662%	0.9414	1.620%	-0.16%	1.464%	0.9395	5	4.71	4.70
10	0.766%	0.9251	1.761%	-0.20%	1.561%	0.9250	105	97.14	97.13
							Bond Price	**141.39**	**141.20**

Table: **Computing the Bond DV01 via slope shift in the forward rates**. 1. Given the current discount curve price the bond (141.39). 2. Given the current discount curve, compute the forward curve, shift it via a slope change, recompute the discount curve and reprice the bond (141.20). 3. The bond DV01 is 0.19=-(141.20-141.39).

Example (The original and shifted forward curve)

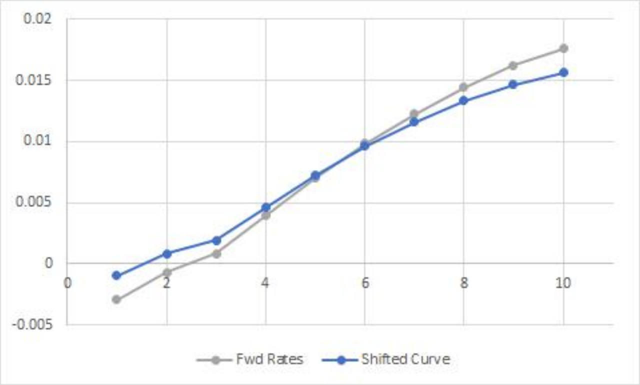


Figure: Slope shift in the forward curve

Example (The initial and shifted value of the bond)

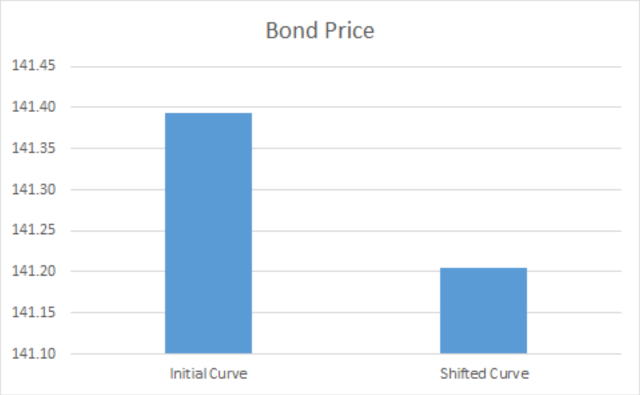


Figure: Slope shift in the forward curve and bond value

Change in the convexity

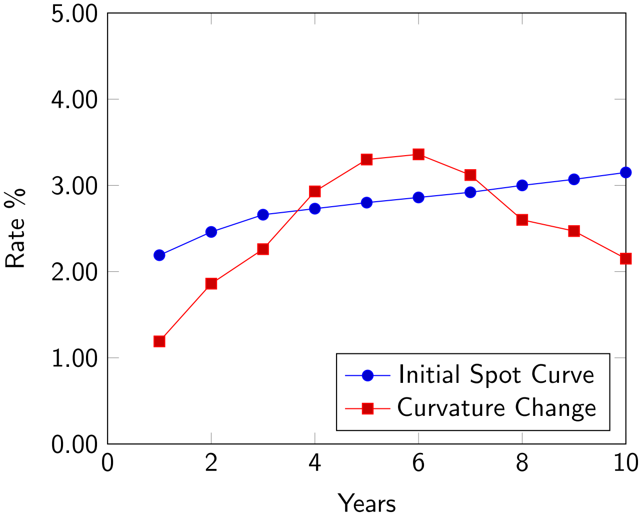


Figure: Change in the curvature of the term structure

Building non parallel shift: Convexity Change

- A curvature shift can be captured by a shift function that is quadratic in the time to maturity

$$\Delta_i = \alpha + \beta \cdot (T_i - t) + \gamma \cdot (T_i - t)^2$$

where, we select three maturities $(T_1 < T_m < T_n)$, and we set

$$\alpha + \beta \cdot (T_1 - t) + \gamma \cdot (T_1 - t)^2 = +\Delta,$$

and

$$\alpha + \beta \cdot (T_m - t) + \gamma \cdot (T_m - t)^2 = -\Delta,$$

and

$$\alpha + \beta \cdot (T_n - t) + \gamma \cdot (T_n - t)^2 = +\Delta.$$

Immunization against ~~both~~ risks:

$$\begin{cases} -0.1181 + 0.00096\, n_1 + \quad n_2 = 0 \\ -0.1884 + 0.000351\, n_1 + \quad n_2 = 0 \end{cases}$$

$$\Delta B \simeq -0.1181\, \Delta^P RF - 0.1884\, \Delta^S RF$$

$$\Delta IRS^{10} \simeq 0.00096\, \Delta^P RF + 0.000351\, \Delta^S RF$$

$$\Delta IRS^{5} \simeq \quad \Delta^P RF + \quad \Delta^S RF$$

$$\Pi = B + n_1 IRS^{10} + n_2 IRS^{5}$$

$$\Delta \Pi = (-0.1181 + n_1 \times 0.00096 + n_2 \times \quad)\Delta^P RF + (-0.1884 + n_1 \times 0.000351 + n_2 \quad)\Delta^S RF$$

Example (Building the curvature change)

- We set $T_1 = 1$, $T_m = 5$ and $T_n = 10$, $t = 0$.
- We set $\Delta_1 = 0.20\%$, $\Delta_5 = -0.20\%$ and $\Delta_{10} = 0.20\%$.
- We find $\alpha = 0.4000\%, \beta = -0.2200\%, \gamma = 0.0200\%$.
- The shift for a generic maturity is given by

$$\Delta_i = 0.4000\% + (-0.2200\%) \cdot T_i + 0.0200\% \cdot T_i^2.$$

Tenor	Swap	DF	Fwd Rates	Curvature Shift	Shifted	DF Shift	CF	PV CF	PV CF (Shift)
		1				1			
1	-0.296%	1.0030	-0.296%	**0.20%**	-0.096%	1.0010	5	5.01	5.00
2	-0.184%	1.0037	-0.072%	0.04%	-0.032%	1.0013	5	5.02	5.01
3	-0.095%	1.0029	0.083%	-0.08%	0.003%	1.0013	5	5.01	5.01
4	0.027%	0.9989	0.395%	-0.16%	0.235%	0.9989	5	4.99	4.99
5	0.160%	0.9920	0.697%	**-0.20%**	0.497%	0.9940	5	4.96	4.97
6	0.295%	0.9824	0.982%	-0.20%	0.782%	0.9862	5	4.91	4.93
7	0.425%	0.9704	1.226%	-0.16%	1.066%	0.9758	5	4.85	4.88
8	0.548%	0.9567	1.442%	-0.08%	1.362%	0.9627	5	4.78	4.81
9	0.662%	0.9414	1.620%	0.04%	1.660%	0.9470	5	4.71	4.74
10	0.766%	0.9251	1.761%	**0.20%**	1.961%	0.9288	105	97.14	97.52
							Bond Price	**141.39**	**141.86**

Table: **Computing the Bond DV01 via slope shift in the forward rates**. 1. Given the current discount curve price the bond (141.39). 2. Given the current discount curve, compute the forward curve, shift it via a change in the curvature, recompute the discount curve and reprice the bond (141.86). 3. The bond DV01 is -0.4714=-(141.86-141.39).

Example

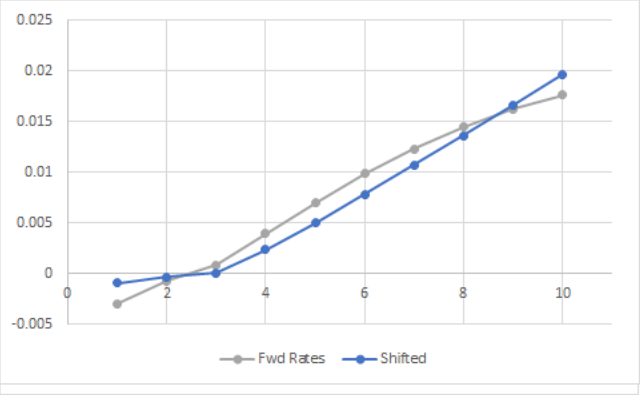


Figure: Initial and shifted forward curve after a change in the curvature

Example

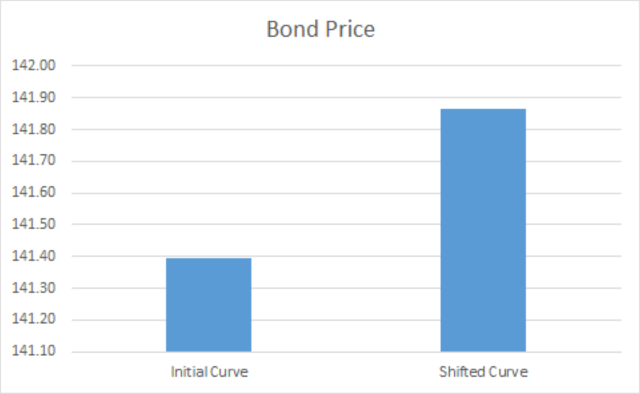


Figure: Initial and shifted bond value after a change in the curvature of the forward curve

Functional risk measures I

- According to this approach we use a term structure model and we examine the sensitivity of the different products to changes in the parameters
- For example, if we adopt the Nelson-Siegel model, we can examine the sensitivity to β_0 (proxy of a change in the level), to β_1 (proxy of a change in the slope), (proxy of a change in the level)
- In this model, the discount curve is assumed to be as follows

$$P_{NS}\left(t, t+\tau; \boldsymbol{\theta}\right) = \exp\left(-\tau \times R_{NS}\left(t, t+\tau; \boldsymbol{\theta}\right)\right).$$

where the continuously compounded spot rate is defined according to

$$R_{NS}\left(t, t+\tau; \boldsymbol{\theta}\right) = \beta_0 + \left(\beta_1 + \frac{\beta_2}{k}\right)\frac{1-\exp\left(-\tau k\right)}{\tau k} - \frac{\beta_2}{k}\exp\left(-\tau k\right),$$

Functional risk measures II

- β_0 specifies the long rate to which the spot rate tends asymptotically
- β_1 is the weight attached to the short term component (spread short/long-term)

$$\lim_{\tau \to 0} R_{NS}(t, t + \tau; \boldsymbol{\theta}) = \beta_0 + \beta_1.$$

- β_2 is the weight attached to the medium term component (a kind of curvature measure).
- k measures the point of the beginning of decay.

Functional risk measures III

We proceed as follows

1. Fit the Nelson-Siegel model and obtain estimated values of β_0, β_1 and β_2 and κ.
2. Price the portfolio using the fitted curve and obtain $\pi(\beta_0, \beta_1, \beta_2, \kappa)$
3. Apply a level shift to the curve by changing $\beta_0 \rightarrow \beta_0 + \Delta\beta_0$
4. Reprice the portfolio and estimate the DV01 with respect to a level shift

$$DV01^{\beta_0} = -(\pi(\beta_0 + \Delta\beta_0, \beta_1, \beta_2, \kappa) - \pi(\beta_0, \beta_1, \beta_2, \kappa))$$

5. Apply a slope shift to the curve by changing $\beta_1 \rightarrow \beta_1 + \Delta\beta_1$
6. Reprice the portfolio and estimate the DV01 with respect to a slope shift

$$DV01^{\beta_1} = -(\pi(\beta_0, \beta_1 + \Delta\beta_1, \beta_2, \kappa) - \pi(\beta_0, \beta_1, \beta_2, \kappa))$$

7. Apply a curvature shift to the curve by changing $\beta_2 \rightarrow \beta_2 + \Delta\beta_2$
8. Reprice the portfolio and estimate the DV01 with respect to a curvature shift

$$DV01^{\beta_2} = -(\pi(\beta_0, \beta_1, \beta_2 + \Delta\beta_2, \kappa) - \pi(\beta_0, \beta_1, \beta_2, \kappa))$$

See Excel File: **FI_InterestRateRisk.xlsm**

Sheet: **Functional Hedging NS**

Example (Level Risk in the Nelson-Siegel Model)

Tenor	Swap	DF	Rates	Spot NS	DF NS	New Rates	New DF	CF	PV CF	PV CF (Shift)
		1			1					
1	-0.296%	1.0030	-0.296%	-0.309%	1.0031	-0.299%	1.0030	5	5.02	5.01
2	-0.184%	1.0037	-0.184%	-0.192%	1.0039	-0.182%	1.0036	5	5.02	5.02
3	-0.095%	1.0029	-0.095%	-0.074%	1.0022	-0.064%	1.0019	5	5.01	5.01
4	0.027%	0.9989	0.027%	0.046%	0.9982	0.056%	0.9978	5	4.99	4.99
5	0.160%	0.9920	0.161%	0.167%	0.9917	0.177%	0.9912	5	4.96	4.96
6	0.295%	0.9824	0.297%	0.290%	0.9828	0.300%	0.9822	5	4.91	4.91
7	0.425%	0.9704	0.429%	0.414%	0.9714	0.424%	0.9707	5	4.86	4.85
8	0.548%	0.9567	0.555%	0.541%	0.9577	0.551%	0.9569	5	4.79	4.78
9	0.662%	0.9414	0.673%	0.669%	0.9416	0.679%	0.9407	5	4.71	4.70
10	0.766%	0.9251	0.781%	0.798%	0.9233	0.808%	0.9223	105	96.94	96.85
			SSE	0.000%				**Bond Price**	**141.20**	**141.09**

	Initial	Shift	New
β_0	-0.8853%	0.01%	-0.8753%
β_1	0.4618%	0.00%	0.4618%
β_2	0.2234%	0.00%	0.2234%
κ	-1.0360%	0.00%	-1.0360%
$\pi(t)$	141.20		141.09

Example (Slope Risk in the Nelson-Siegel Model)

Tenor	Swap	DF	Rates	Spot NS	DF NS	New Rates	New DF	CF	PV CF	PV CF (Shift)
		1			1					
1	-0.296%	1.0030	-0.296%	-0.309%	1.0031	-0.299%	1.0030	5	5.02	5.01
2	-0.184%	1.0037	-0.184%	-0.192%	1.0039	-0.182%	1.0036	5	5.02	5.02
3	-0.095%	1.0029	-0.095%	-0.074%	1.0022	-0.064%	1.0019	5	5.01	5.01
4	0.027%	0.9989	0.027%	0.046%	0.9982	0.056%	0.9978	5	4.99	4.99
5	0.160%	0.9920	0.161%	0.167%	0.9917	0.177%	0.9912	5	4.96	4.96
6	0.295%	0.9824	0.297%	0.290%	0.9828	0.300%	0.9822	5	4.91	4.91
7	0.425%	0.9704	0.429%	0.414%	0.9714	0.425%	0.9707	5	4.86	4.85
8	0.548%	0.9567	0.555%	0.541%	0.9577	0.551%	0.9569	5	4.79	4.78
9	0.662%	0.9414	0.673%	0.669%	0.9416	0.679%	0.9407	5	4.71	4.70
10	0.766%	0.9251	0.781%	0.798%	0.9233	0.809%	0.9223	105	96.94	96.84
			SSE	0.000%				**Bond Price**	**141.20**	**141.08**

	Initial	Shift	New
β_0	-0.8853%	0.00%	-0.8853%
β_1	0.4618%	0.01%	0.4718%
β_2	0.2234%	0.00%	0.2234%
κ	-1.0360%	0.00%	-1.0360%
$\pi(t)$	141.20		141.08

Example (Curvature Risk in the Nelson-Siegel Model)

Tenor	Swap	DF	Rates	Spot NS	DF NS	New Rates	New DF	CF	PV CF	PV CF (Shift)
		1			1					
1	-0.296%	1.0030	-0.296%	-0.309%	1.0031	-0.304%	1.0030	5	5.02	5.02
2	-0.184%	1.0037	-0.184%	-0.192%	1.0039	-0.182%	1.0036	5	5.02	5.02
3	-0.095%	1.0029	-0.095%	-0.074%	1.0022	-0.059%	1.0018	5	5.01	5.01
4	0.027%	0.9989	0.027%	0.046%	0.9982	0.066%	0.9974	5	4.99	4.99
5	0.160%	0.9920	0.161%	0.167%	0.9917	0.193%	0.9904	5	4.96	4.95
6	0.295%	0.9824	0.297%	0.290%	0.9828	0.321%	0.9809	5	4.91	4.90
7	0.425%	0.9704	0.429%	0.414%	0.9714	0.451%	0.9689	5	4.86	4.84
8	0.548%	0.9567	0.555%	0.541%	0.9577	0.583%	0.9544	5	4.79	4.77
9	0.662%	0.9414	0.673%	0.669%	0.9416	0.717%	0.9375	5	4.71	4.69
10	0.766%	0.9251	0.781%	0.798%	0.9233	0.852%	0.9183	105	96.94	96.42
			SSE	0.000%				**Bond Price**	**141.20**	**140.61**

	Initial	Shift	New
β_0	-0.8853%	0.00%	-0.8853%
β_1	0.4618%	0.00%	0.4618%
β_2	0.2234%	0.01%	0.2334%
κ	-1.0360%	0.00%	-1.0360%
$\pi(t)$	141.20		140.61

Functional Hedging in the Nelson-Siegel Model I

Example (Step 1: Pricing)

	Bond	IRS1	IRS2	Portfolio		Quantities
β_0	0.110	0.020	0.096	0.11	Bond	1.000
β_1	0.120	0.020	0.102	0.12	IRS1	0.000
β_2	0.590	0.020	0.509	0.59	IRS2	0.000

Table: Exposures (measured via DV01) of the unhedged portfolio. IRS1: fixed rate swap with 2 years tenor; IRS2: fixed rate swap with 10 years tenor

Functional Hedging in the Nelson-Siegel Model II

Example (Step 2: Hedging level risk)

	Bond	IRS1	IRS2	Portfolio		Quantities
β_0	0.110	0.020	0.096	0	Bond	1.000
β_1	0.120	0.020	0.102	0.01	IRS1	-5.500
β_2	0.590	0.020	0.509	0.48	IRS2	0.000

Table: Hedging Level Risk. IRS1: fixed rate swap with 2 years tenor; IRS2: fixed rate swap with 10 years tenor

Choose quantities such that

$$0.110 + 0.020 \cdot n_1 = 0$$

This gives

$$n_1 = -5.5, \quad n_2 = 0$$

Functional Hedging in the Nelson-Siegel Model III

Example (Step 3: Hedging slope risk)

	Bond	IRS1	IRS2	Portfolio		Quantities
β_0	0.110	0.020	0.096	-0.0029	Bond	1.000
β_1	0.120	0.020	0.102	0.0000	IRS1	0.000
β_2	0.590	0.020	0.509	-0.0088	IRS2	-1.176

Table: Hedging Slope Risk. IRS1: fixed rate swap with 2 years tenor; IRS2: fixed rate swap with 10 years tenor

Choose quantities such that

$$0.120 + 0.102 \cdot n_2 = 0$$

This gives

$$n_1 = 0, \quad n_2 = -1.176$$

Functional Hedging in the Nelson-Siegel Model IV

Example (Step 4: Hedging Level & Slope risk)

	Bond	IRS1	IRS2	Portfolio		Quantities
β_0	0.110	0.020	0.096	0.0000	Bond	1.000
β_1	0.120	0.020	0.102	0.0000	IRS1	2.500
β_2	0.590	0.020	0.509	-0.2083	IRS2	-1.667

Table: Hedging Level & Slope Risk. IRS1: fixed rate swap with 2 years tenor; IRS2: fixed rate swap with 10 years tenor

Choose a number n_1 and n_2 of the two swaps such that

$$0.110 + 0.02 \cdot n_1 + 0.096 \cdot n_2 = 0$$

and

$$0.120 + 0.02 \cdot n_1 + 0.102 \cdot n_2 = 0$$

This gives

$$n_1 = 2.5, \quad n_2 = -1.667$$

Extending Functional Hedging

- The approach used with the Nelson-Siegel model can be extended to other models.
- For example, if we adopt the Vasicek model we can consider hedging against changes
  - in the short rate $r(t)$
  - in the long-run mean $\mu(t)$
  - in the volatility parameter σ
  - in the mean-reversion speed α
- Similar considerations can be done for other models

Convenient Risk Factors

- Often, it's natural to define an appropriate risk factor for each contract.
- Example 1. Forward Rate Agreement.
- Example 2. Interest Rate Swap.
- Example 3. Floating Rate Note.
- Example 4. Caplet.
- Example 5. Swaption.

Example (DV01 of a Forward Rate Agreement)

- A FRA contract with reset in T_1 and payment in T_2, and fixed rate F has a fair value equal to

$$FRA(t) = P(t, T_2)\left(F(t, T_1, T_2) - F\right)\alpha_{T_1,T_2},$$

- According to this formula, a FRA can take positive or negative values depending on the change in the forward rate.
- We take it as risk factor, so that

$$\boxed{\frac{\partial FRA(t)}{\partial F(t, T_1, T_2)} = P(t, T_2) \times \alpha_{T_1,T_2},}$$

- Therefore

$$\boxed{DV01_{FRA} = -P(t, T_2) \times \alpha_{T_1,T_2} \times \frac{1}{10,000}}$$

Example (DV01 of a Floating Rate Note)

- A FRN with current coupon c and next coupon date in T_1 is worth

$$FRN(t) = P(t, T_1) \times (1 + c \times \alpha_{T_0,T_1}).$$

- The FRN price moves, due to changes in $P(t, T_1)$. These changes are driven by the T_1 spot rate, a LIBOR rate in general, that it is our risk factor so we write

$$FRN(t) = \frac{1 + c \times \alpha_{T_0,T_1}}{1 + L(t, T_1)\alpha_{t,T_1}},$$

and we have

$$\frac{\partial FRN(t)}{\partial L(t, T_1)} = -\alpha_{t,T_1} \times \frac{1 + c \times \alpha_{T_0,T_1}}{(1 + L(t, T_1)\alpha_{t,T_1})^2}.$$

- Therefore

$$DV01_{FRN} = \alpha_{t,T_1} \times P^2(t, T_1) \times (1 + c \times \alpha_{T_0,T_1}) \times \frac{1}{10,000}.$$

Example (DV01 of an Interest Rate Swap)

- An IRS having: a) reset dates in $T_0, T_1, \cdots, T_{n-1}$ with $T_0 > t$, b) payment dates in $T_0, T_1, \cdots, T_{n-1}$, c) fixed swap rate S, has a fair value equal to

$$Swap(t) = (S(t, T_0, T_n) - S) \times Annuity(t, T_0, T_n),$$

where

$$Annuity(t, T_0, T_n) = \sum_{i=1}^{n} P(t, T_i) \times \alpha_{T_{i-1}, T_i}.$$

- According to this formula, a swap can take positive or negative values depending on the change in the forward swap rate.
- We take it as risk factor, so that

$$\frac{\partial Swap(t)}{\partial S(t, T_0, T_n)} = Annuity(t, T_0, T_n)$$

and then

$$DV01_{FRA} = -Annuity(t, T_0, T_n) \times \frac{1}{10,000}$$

- A modified formula is needed if the payment in T_1 is already known (i.e. $T_0 < t < T_1$), so $Annuity(t, T_0, T_n) = \sum_{i=\mathbf{2}}^{n} P(t, T_i) \times \alpha_{T_{i-1}, T_i}$.

Example (DV01 of a Caplet in the Black model)

- A Caplet contract with reset in T_1 and payment in T_2, and strike K is priced according to to

$$Caplet(t) = P(t, T_2) \times (F(t, T_1, T_2)N(d_1) - KN(d_2)) \times \alpha_{T_1,T_2}.$$

- According to this formula, a caplet is exposed to changes in the forward rate.
- We take it as risk factor, so that

$$\boxed{\frac{\partial Caplet(t)}{\partial F(t, T_1, T_2)} = P(t, T_2) \times N(d_1) \times \alpha_{T_1,T_2}.}$$

- Therefore

$$\boxed{DV01_{Caplet} = -P(t, T_2) \times N(d_1) \times \alpha_{T_1,T_2} \times \frac{1}{10,000},}$$

  i.e. the DV01 of a FRA contract multiplied by the caplet delta.
- This formula assumes that the Black formula is used. If we adopt a different pricing model, e.g. Bachelier, the DV01 will also change.

A Case Study

1. Consider portfolio constituents
2. Measure interest rate risk of each componet
3. Compute portfolio exposure according to a given risk factor
4. Adjust portfolio composition to hedge againt the risk factor

Measuring the Price Sensitivity of Portfolios

- We describe how measures of portfolio price sensitivity are related to the measures of its component securities.
- We have $i = 1,\ldots,\ n$ interest rated derivatives (e.g. coupond bonds) with prices B_i, and risk exposures $DV01_i$ and we have an amount a_i invested in each.
- By definition, the value of a portfolio equals the sum of the values of the individual securities in the portfolio

$$\pi = \sum_{i=1}^{n} a_i B_i.$$

- We can compute the DV01 of the portfolio $DV01_\pi$ as sum of the individual DV01 values (this is possible because the DV01's are money amounts)

$$DV01_\pi = \sum_{i=1}^{n} a_i DV01_i.$$

Hedging a portfolio of bonds using DV01

1. Hedging a portfolio

- We are hedged against interest rate risk if we choose the amounts a_i such that

  DV01 constraint

$$\sum_{i=1}^{n} a_i \times DV01_i = 0,$$

  and

  Balance constraint

$$\sum_{i=1}^{n} a_i \times B_i = \pi.$$

2. Setting up a linear system

- This amounts to solve the $2 \times n$ linear system with respect to the n unknowns $a_1, ..., a_n$:

$$\begin{bmatrix} DV01_1 & DV01_2 & ... & DV01_n \\ B_1 & B_2 & ... & B_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \end{bmatrix}$$

- In general, the linear system admits an infinite number of solutions.

Example (1. Computing Portfolio Exposure)

The calculations details can be found in the Excel file BasicYields.xlsm, Sheet: Hedging a Portfolio

- Let us suppose we have the following portfolio

Bond	Gross Price	Yield %	FV	Units	Value	$DV01 \times 10,000$
1	99.9	2.35	100	11	1098.9	2.44
2	101.2	2.45	100	-12	-1214.4	3.19
3	103.32	2.8	100	7	723.24	4.34
Portfolio					**607.74**	**18.95**

- In particular, we have

$$DV01_{\pi} \times 10000 = 11 \times 2.44 - 12 \times 3.19 + 7 \times 4.34 = 18.95.$$

Example ((...ctd) 2. Changes in the term structure and change in the portfolio value)

Let us assume that changes in the YTM are perfectly correlated and same volatility. Therefore,

- if YTMs move up by 100 basis points, the change in the portfolio value is estimated as

$$\Delta\pi = -18.95 \times 1\% = -1.895,$$

  so the new value of the portfolio is

$$607.74 - 1.895 = 605.845.$$

- Similarly, if the term structure moves down by 100 basis points, the new portfolio value is

$$607.74 + 1.895 = 609.635.$$

Example ((...ctd) 3. Immunize the portfolio: solve a linear system)

- We would like to change the portfolio composition so that we are immunized (hedged) against term structure movements.
- We have to find the quantities a_i to be invested in each bond. We have to solve the following linear system

$$\begin{bmatrix} 2.44 & 3.19 & 4.34 \\ 99.9 & 101. & 103.32 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 607.74 \end{bmatrix}$$

- The Linear System has 3 unknowns and 2 equations: there is an infinite number of solutions.
- We decide (arbitrarily) to keep unchanged the quantity of the second bond ($a_2 = -12$).
- Setting $a_2 = -12$, the linear system becomes

$$\begin{bmatrix} 2.44 & 4.34 \\ 99.9 & 103.32 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12 \times 3.19 \\ 607.74 + 12 \times 101 \end{bmatrix} = \begin{bmatrix} 38.28 \\ 1819.74 \end{bmatrix}$$

Example ((...ctd) 4. The new portfolio composition)

- We solve the linear system and we get

$$a_1 = 21.78, a_2 = -12, a_3 = -3.42.$$

- The new portfolio compostion is

	Mkt Gross Price	Yield %	Nominal	Quantity	Value	DV01
BTP 1	99.9	2.35	100	**21.78**	2175.63	2.44
BTP 2	101.2	2.45	100	**-12.00**	-1214.4	3.19
BTP 3	103.32	2.8	100	**-3.42**	-353.49	4.34
					607.74	**0.00**

- Notice that
  - The portfolio value is unchanged, due to the balance constraint;
  - The portfolio exposure to term structure shifts is now 0.
- Whatever the term structure shift (provided it is not too large and of parallel type), the portfolio value will not be affected.

Question

$B^1 = 99 \qquad B^2 = 100$

$DV01^1 = -0.2 \qquad DV01^2 = -0.8$

You hold 10 and 20 units of two bonds. Their market prices are 99 and 100. Their DV01s are -0.2 and -0.8.

- Compute the portfolio DV01; $DV01^{\Pi} = 10 \times (-0.2) + 20 \times (-0.8) = -18$
- How much do you lose or gain if there is an up movement of 20 bp in the market rates; $\Delta y = 20\text{bps} \qquad \Delta\Pi = DV01^{\Pi} \times \Delta y = -18 \times 20 = 360$
- You are considering to buy a zero-cost derivative (eg a swap), having DV01 equal to 0.6. How many swaps do you need to hedge the exposure of your portfolio?

$$DV01^{SWAP} = 0.6$$

$$\Delta\Pi = \left(DV01^{\Pi} + n \times DV01^{SWAP}\right)\Delta y = 0$$

$$-18 + 0.6n = 0$$

$$n = 30$$

Answer

Conclusions

- DV01 is a simple risk analytics easy to comunicate and to compute (in general).
- Term structure does not move in a parallel way.
- Different points of the term structure have different volatility.
- There is a large correlation among changes in the risk-factors.
- We have seen how to manage different kind of interest rate changes
- Another procedure is via Principal Component Analysis
- Principal Component Analysis is a statistical procedure that allows to take into account non perfect correlation in the term structure changes via a factor model.

Appendix

Duration

- Duration hedging of a portfolio is based on a single risk variable, the yield to maturity of the portfolio.

Macaulay duration

The `Macaulay bond duration` is measured as the weighted maturity of each payment, where the weights are proportional to the present value of the cash flows, if these are predetermined

$$D = \frac{\sum_{i=1}^{n} \frac{(t_i - t) \times c/m}{(1+y)^{t_i - t}} + \frac{(t_n - t) \times 100}{(1+y)^{t_n - t}}}{\sum_{i=1}^{n} \frac{c/m}{(1+y)^{t_i - t}} + \frac{100}{(1+y)^{t_n - t}}}.$$

Textbook Example: Duration of a BTP

	B	C	D	E
3	**Example 1: BTP**			
4				
5	Yield to maturity		0.028581586	
6	Coupon (annual)		0.03	
7	Frequency		2	
8	Time since last coupon		0.27	
9	Fraction of coupon period		0.54	
10	Accrued Interest		2	
11	Clean Price		100.69	
12	Gross Price		101.23	
13	Payment Dates (years)	Coupons	$PV(c_k)\ t_k$	
14	0.23	1.5	0.3428	=C14*B14/(1+$D$5)^B14
15	0.73	1.5	1.0727	=C15*B15/(1+$D$5)^B15
16	1.23	1.5	1.7821	=C16*B16/(1+$D$5)^B16
17	1.73	1.5	2.4715	=C17*B17/(1+$D$5)^B17
18	2.23	1.5	3.1413	=C18*B18/(1+$D$5)^B18
19	2.73	101.5	256.5765	=C19*B19/(1+$D$5)^B19
20	**Duration (years)**		**2.6216**	=SOMMA(D14:D19)/D12

Textbook Example: Duration of a Treasury Bond

	B	C	D	E
24	**Example 2: Treasury Bond**			
25				
26	Yield to maturity		0.028380223	
27	Coupon (annual)		0.03	
28	Frequency		2	
29	Time since last coupon		0.27	
30	Fraction of coupon period		0.54	
31	Accrued Interest		2	
32	Clean Price		100.69	
33	Gross Price		101.23	
34	Payment Dates (semesters)	Coupons	$PV(c_k)\ t_k$	
35	0.46	1.5	0.6855	=C35*B35/(1+$D$26/2)^B35
36	1.46	1.5	2.1454	=C36*B36/(1+$D$26/2)^B36
37	2.46	1.5	3.5643	=C37*B37/(1+$D$26/2)^B37
38	3.46	1.5	4.9430	=C38*B38/(1+$D$26/2)^B38
39	4.46	1.5	6.2825	=C39*B39/(1+$D$26/2)^B39
40	5.46	101.5	513.1530	=C40*B40/(1+$D$26/2)^B40
41	Duration (semesters)		5.2432	=SOMMA(D35:D40)/D33
42	**Duration (years)**		**2.6216**	=D41/2

Computing the Duration in Excel

	A	B	C	D	E	F	G	H
1			Duration of a BTP					
2	BTP 01.02.2006							
3	ISIN	IT0003424485						
4	Maturity	2/1/2006						
5	Trade Date	10/28/2003						
6	Value Date	10/31/2003	Cell B6 has been named ValueDate					
7	Payment of last coupon	8/1/2003	=Couppcd(ValueDate,B4,2)					
8	Date of next coupon	2/1/2004	=Coupncd(ValueDate,B4,2,1)					
9	Annual coupon	2.75			=SE(B20=1,A20+1,SE(B20=7,A20+2,A20))			
10	Days since last coupon	91	=B6-B7			=C21-ValueDate		
11	Days to next coupon	184	=B8-B7					
12	Clean Price	99.76000						
13	Accrued Interest	0.68003	=(B9/2)*B10/(B11)					
14	Market Gross Price	100.44003	=B12+B13					
15	Yield to Maturity	**2.8738%**				=$B$9/2		
16		Week Day						
17	Coupon Dates	1 (sunday) to 7 (saturday).	Corrected	days	Semiannual	c_k * PV(t_k)	t_k * PV(t_k)	
18			Coupon Dates		Coupon c_k			
19	2/1/2004	1	2/2/2004	94	1.375	1.36500	0.351535209	=D19*F19/365
20	8/1/2004	1	8/2/2004	276	1.375	1.34586	1.017687897	=D20*F20/365
21	2/1/2005	3	2/1/2005	459	1.375	1.32687	1.66858776	=D21*F21/365
22	8/1/2005	2	8/1/2005	640	1.375	1.30836	2.294111938	=D22*F22/365
23	2/1/2006	4	2/1/2006	824	1.375	1.28981	2.91178277	=D23*F23/365
24	2/1/2006	4	2/1/2006	824	100	93.80412	211.7660196	=D24*F24/365
25					Sum	100.44	220.01	
26		=G25/F25			Duration	2.19046		
27		=F26/(1+B15)			Modified Duration	2.12927		
28								

Duration as sensitivity measure

Duration and interest rate exposure

Duration is a measure of interest-rate exposure to parallel shifts in the yield to maturity

$$\frac{\partial B\left(y; t, t_1, \ldots, t_n\right)}{\partial y} = -\frac{D}{(1+y)} B = -D_M \times B.$$

where $D_M = \frac{D}{(1+y)}$ is `modified duration`. The term $D_M B$ is known as `dollar duration`.

- The discrete variation in the bond price given a discrete variation in the yield can be approximated as

$$\begin{aligned} \text{absolute P\&L: } \Delta B &\simeq -D_M \times B \times \Delta y, \\ \text{relative P\&L: } \frac{\Delta B}{B} &\simeq -D_M \times \Delta y. \end{aligned}$$

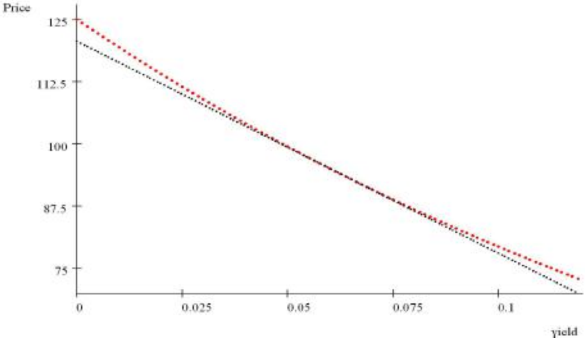
Duration: a linear approximation to the price function

- Using a one-order Taylor expansion:

$$B\left(y^* + \Delta y\right) \simeq \underbrace{B\left(y^*\right)}_{intercept} - \underbrace{D_M \times B\left(y^*\right)}_{slope} \times \left(y - y^*\right),$$

where y^* is the yield to maturity.

- The approximation works well only for small parallel shifts in the term structure.



Example (Estimating bond price changes)

Let us suppose that

$$B = 100.44,\ y = 2.8733\%,\ D = 2.19046,\ D_M = \frac{D}{1+y} = \frac{2.19046}{1+0.028733} = 2.12927.$$

If we consider a 10 basis point change in the yield to maturity (1bp is 1% of 0.01 or 0.0001), i.e. $dy = 10bp = 10 \times 0.0001$, then

$$\text{absolute P\&L: } \Delta B \simeq -D_M \times B \times \Delta y = -2.12927 \times 100.44 \times 0.001 = -0.21386,$$
$$\text{relative P\&L: } \frac{\Delta B}{B} \simeq -D_M \times \Delta y = 2.12927 \times 0.001 = 0.00212927.$$

This means that an increase in the ytm of 10bp implies an

1. a decrease in the bond price of 0.21386 Euros;
2. the new bond price will be $B^* = B + \Delta B = 100.44 - 0.21386 = 100.2261634;$
3. a percentage variation in the bond price of 0.00212927;

Babcock's formula: Duration as immunization length

Duration gives the neutral investment horizon

Babcock's formula

The Babcock's formula gives a simple (approximate) expression for the holding period return in terms of ω the holding period, y the YTM, D_0 the Duration at the beginning of the investment period and Δy the change in the YTM:

$$y_\omega = y - \frac{D_0 - \omega}{\omega} \Delta y.$$

- The holding period return (return ex-post) is equal to the YTM (return ex-ante) if
  1. there are no variations in the YTM, i.e. $\Delta y = 0$, or if
  2. the holding period is equal to the Duration.
- The risk of having an ex-post return different from y is larger, greater the difference between $D_0 - \omega$ (excess Duration).

Hedging a portfolio of bonds using Duration

- Let us assume that the term structure of ytm is flat and that it is subject to a parallel shift, i.e. approximately

$$\Delta y_1 \simeq \Delta y_2 \simeq ... \simeq \Delta y_n \simeq \Delta y.$$

- Then, using the fact that $DV01_i = D_{M_i} \times B_i / 10000$, we have

$$DV01_\pi = \sum_{i=1}^{n} a_i \times D_{M_i} \times \frac{B_i}{10000}.$$

- If we express the DV01 of the portfolio in terms of the individual durations, we get

$$\frac{D_\pi \times B_\pi}{(1 + y_\pi) \times 10000} = \sum_{i=1}^{n} a_i \times D_i \times \frac{B_i}{(1 + y_i) \times 10000}.$$

Portfolio duration as sensitivity measure

- Simplifying, and assuming that $y_1 = ... = y_n = y_\pi$, we obtain a convenient property.

Portfolio Duration

Under the assumption that the term structure of yields is flat, the duration of a portfolio is the weighted average of each bond's duration

$$D_\pi = \sum_{i=1}^{n} w_i D_i,$$

where

$$w_i = \frac{a_i \times B_i}{B_\pi} = \frac{a_i \times B_i}{\sum_{j=1}^{n} B_j}.$$

Hedging a portfolio of bonds

- The portfolio value is then subject to a change given by

$$\Delta\pi = \sum_{i=1}^{n} a_i \Delta B_i = -\left(\sum_{i=1}^{n} a_i \times D_{M,i} \times B_i\right) \Delta y.$$

- We are hedged against interest rate risk if we rebalance the amounts a_i such that

duration constraint

$$\sum_{i=1}^{n} a_i \times D_{M,i} \times B_i = 0,$$

and

balance constraint

$$\sum_{i=1}^{n} a_i \times B_i = \pi.$$

- At the end, we still have to solve a linear system (see the Excel file).

The multi-factor approach I

- We can compute the DV01 of each bond with respect to each point of the term structure, say K different points.
- Therefore, at the first order we can write

$$\Delta B_i = \sum_{j=1}^{K} DV01_{i,j} \Delta f_j$$

- The portfolio change is described by

$$\Delta P\&L = \sum_{i=1}^{N} q_i \Delta B_i = \sum_{i=1}^{N} q_i \sum_{j=1}^{K} DV01_{i,j} \Delta f_j.$$

The multi-factor approach II

- We can rewrite the above as follows

$$\Delta P\&L = \sum_{j=1}^{K} \beta_j \Delta f_j,$$

where

$$\beta_j = \sum_{i=1}^{N} q_i DV01_{i,j}.$$

- We can interpret those coefficients as

$$\beta_j = \text{Portfolio Exposure to factor j}, j = 1, ..., K$$

The multi-factor approach III

- A possibility to manage this exposure is to choose the quantities such that the profit and loss variance is minimized:

$$\min_{q} var(\Delta P\&L)$$

  always under a budget constraints, i.e. the portfolio value does not change before and after the change in the allocation.

- In this case, we try to exploit in an efficient way the large correlation among risk factors.

Example (Building a Minimum variance hedge)

- Let us suppose that we would like to hedge the exposure of a portfolio of bonds by using a number n of a zero-cost interest rate swap, so that

$$\pi_{hedged} = \pi_{unhedged} + n \times 0.$$

- The exposure of the hedged portfolio is

$$\Delta\pi_{hedged} = -DV01_p \Delta f_p - nDV01_{swap}\Delta f_{swap}$$

- If $\Delta f_p = \Delta f_{swap}$, we can perfectly hedge the portfolio by setting

$$\hat{n} = -\frac{DV01_p}{DV01_{swap}}.$$

- If the two risk-factors do not move in a 1-1 relationship, we can try a minimum variance hedge.

A Case Study I

1 Let us consider the following bonds having the exposures to different interest rates as in Table.

Table: Portfolio Characteristics: for example the yellow cells give the exposures of the fourth bond to changes in the 1 year spot rate

Bond	Mkt Price	1	2	3	4	5	Weights
1	100	9.52	3.68	3.52	2.58	5.71	0.23
2	100	4.42	0.26	0.33	2.4	7.41	0.06
3	100	4.31	5.77	2.44	4.27	1.93	0.09
4	100	9.25	3.74	4.03	4.79	1.82	0.21
5	100	1.42	3.18	2.44	6.57	5.41	0.02
6	100	1.3	4.15	3.82	2.99	7.74	0.17
7	100	7.43	3.19	1.56	1.04	4.21	0
8	100	4.24	5.66	4.03	3.52	4.09	0.22

A Case Study II

2 The portfolio exposures to the different interest rates are

Table: Portfolio Exposures: multiply the matrix of the exsposures with the weights vector, eg. $5.9674 = 9.52 \times 0.23 + 3.68 \times 0.06 + \cdots + 5.71 \times 0.22$

	1	2	3	4	5
P&L	5.9674	4.181	3.4801	3.5417	4.6376

3 We have an amount of 100 USD to allocate among the different bonds.

4 The covariance matrix of changes in spot rates is as follows

Table: Covariance matrix of changes in spot rates

1	2	3	4	5
0.001594	0.001826	0.001902	0.001835	0.001755
0.001826	0.002326	0.002507	0.002464	0.002398
0.001902	0.002507	0.002779	0.002783	0.002751
0.001835	0.002464	0.002783	0.002848	0.002864
0.001755	0.002398	0.002751	0.002864	0.002924

A Case Study III

5 We can compute the portfolio variance

$$var(P\&L) = \sum_i \sum_j \beta_j \beta_j \sigma_{i,j}$$

6 Given the current allocation, the variance of the P&L is 1.065.

7 We solve the minimization problem, also imposing constraints on the maximum and minimum weights (20% and 5%).

8 We can set up the optimization problem in Excel.

A Case Study IV

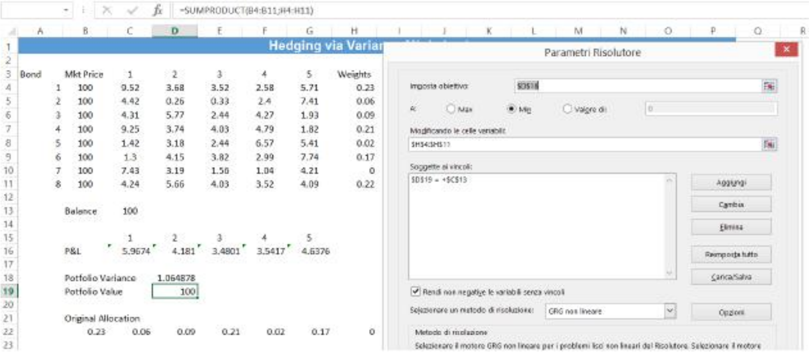


Figure: Setting up the optimization problem in Excel using the Solver

A Case Study V

9 The optimizer suggests the optimal allocation

Table: The minimum variance allocation

Bond	1	2	3	4	5	6	7	8
Original	0.23	0.06	0.09	0.21	0.02	0.17	0	0.22
New	0.05	0.2	0.2	0.05	0.2	0.05	0.2	0.05

10 The new portfolio exposures are

Table: Comparing new and old portfolio

Factor	1	2	3	4	5
Original	5.967	4.181	3.480	3.542	4.638
New	4.7315	3.3415	2.124	3.55	4.76

A Case Study VI

11 We can examine the out-of sample performance of the two portfolios. We achieve a 23% volatility reduction.

Table: Old and new portfolio

	Original	New
Volatility	0.175	0.134
Min return	-1.692	-1.483
Mean return	-0.031	-0.026
Max return	2.700	2.327

Financial Engineering and Bond Replication

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SMM269 Fixed Income

Academic Year 2019-20

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Required Readings

- Any derivatives textbook covering strategies in options.
- J. Hull, Options, Futures, and Other Derivatives, Pearson; 10th edition (January 30, 2017), **Chapter 11. Trading strategies involving options**

Exercises

- Question_Solutions_QM_FI.pdf, Chapter X - Coupon Decomposition

Outline

Basic Payoffs

Basic Payoffs: formula

General Coupon Formula (Reset in Advance, Pay in Arrears)

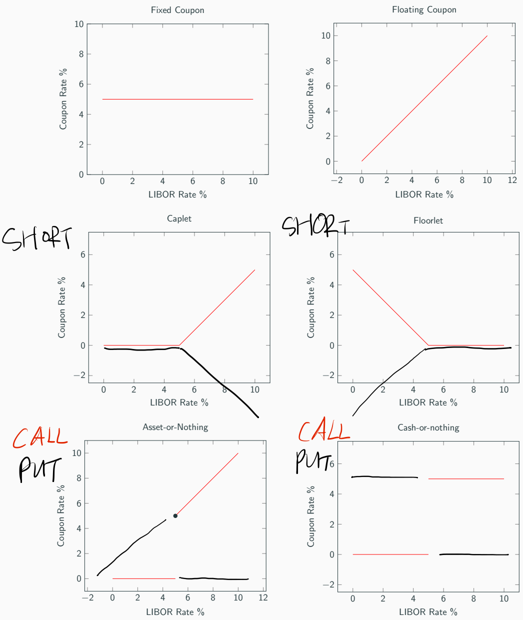
$$c(T_i) = f(RefRate(T_{i-1})) \times \alpha_{T_{i-1},T_i} \times N.$$

Popular Coupon Formula

Fixed Coupon	Floating
$c \times \alpha_{T_{i-1},T_i}$	$L(T_{i-1}, T_i) \times \alpha_{T_{i-1},T_i}$
Caplet $\max(L(T_{i-1}, T_i) - K, 0) \times \alpha_{T_{i-1},T_i}$	**Floorlet** $\max(K - L(T_{i-1}, T_i), 0) \times \alpha_{T_{i-1},T_i}$
Asset or Nothing $L(T_{i-1}, T_i) \times \mathbf{1}_{L(T_{i-1},T_i)>K} \times \alpha_{T_{i-1},T_i}$	**Cash or Nothing** $\mathbf{1}_{L(T_{i-1},T_i)>K} \times \alpha_{T_{i-1},T_i}$

Table 1: The reference rate is observed at the beginning of the coupon period (Reset in Advance), but the payment occurs at the end of the period (Pay in Arrears). In the Table the Face Value is assumed to be 1.

Table 2: Basic Payoffs: diagram



Financial Engineering: Coupon Replication

Problem

- How to use the basic payoffs to build up the coupon of the structured product?
- In general, the basic payoffs can be priced via closed form pricing formulas. This will be the topic of the next lectures.
- Then if we can decompose the coupon payment into basic payoffs (i.e. static replica), then it is easy to price the bond.
- If this is not possible, we need to develop ad hoc pricing formula for the bond.

Coupon Formula i

- Let us consider the payoff in Formula (1)

$$C(T_i) = \begin{cases} 3\% & \textit{if} \quad L(T_{i-1}, T_i) \leq 3\% \\ L(T_{i-1}, T_i) & \textit{if} \quad 3\% \leq L(T_{i-1}, T_i) \leq 5\% \\ 5\% & \textit{if} \quad 5\% \leq L(T_{i-1}, T_i) \end{cases} \qquad (1)$$

- The coupon profile is given in figure 1.

Coupon Formula ii

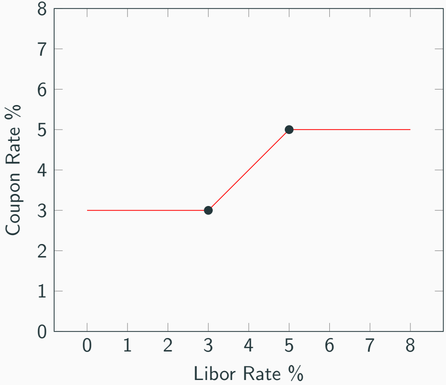
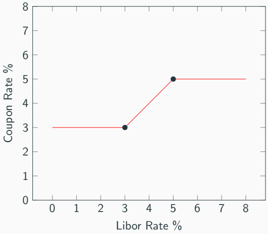


Figure 1: Payoff profile of coupon formula 1

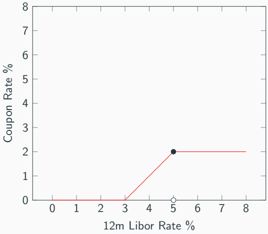
- We want decompose it using basic payoffs.

Coupon Formula iii

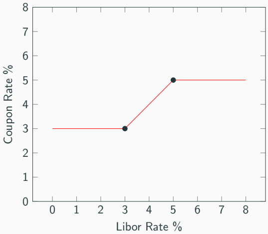
- We can start by identifying a fixed amount equal to 3%



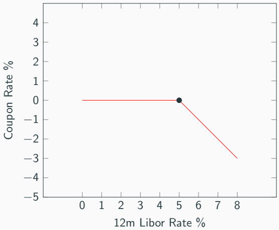
$= 3\% +$



- We can also identify a caplet, struck at 3%.



$$= 3\% + (E - 3\%)^+ +$$



- The last component is a short caplet at 5%.
- Therefore

$$C(T_i) = 3\% + (E - 3\%)^+ - (E - 5\%)^+.$$

Verify the Coupon Replication

Verification Coupon Formula 1

- We can check that our decomposition is correct, considering the four components in two cases: Libor below 3%, between 3% and 5%, above 5%.
- This is done in the following Table

Component	$E < 3\%$	$3\% < E < 5\%$	$E > 5\%$
Fixed	3%	3%	3%
Long Caplet at $K = 3\%$	0	$E - 3\%$	$E - 3\%$
Short Caplet at $K = 5\%$	0	0	$5 - E\%$
Sum	3%	E	5%

Table 3: Verification of the payoff decomposition. First column: different option components. Second to fourt columns: payoffs depending on the value of the Libor rate.

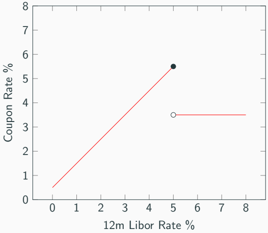
Another example with digital payoffs

Coupon Replication (Decomposition 1)

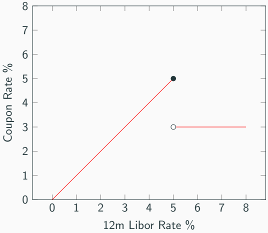
- Let us consider the payoff in Formula (2)

$$C(T_i) = \begin{cases} L(T_{i-1}, T_i) + 50bp \quad if \quad UR(T_{i-1}, T_i) \leq 5\% \\ \\ 3.50\% \qquad\qquad\qquad otherwise \end{cases} \tag{2}$$

- We want decompose it using basic payoffs.
- We can start by identifying a fixed amount equal to 0.5%

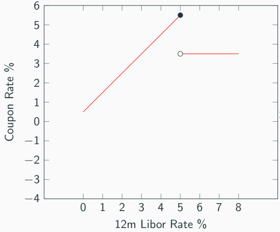


$= 0.5\% +$

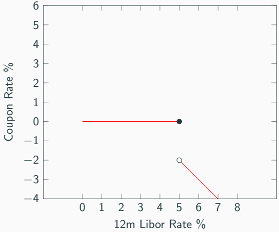


Coupon Replication (Decomposition 2)

- We can continue by identifying a floating amount

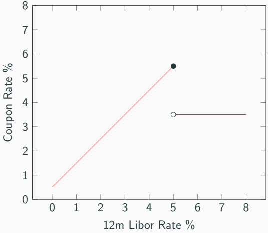


$$= 0.5\% + E +$$

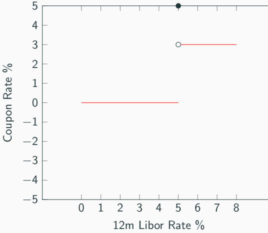


Coupon Replication (Decomposition 3) i

- We can continue by identifying a short asset-or-nothing with strike at 5%



$$= 0.5\% + E - E \times 1_{E>5\%}$$



Coupon Replication (Decomposition 3) ii

- We complete the decomposition by identifying a number 3% of cash or nothing options.
- Therefore

$$coupon = 0.5\% + E - E \times 1_{E>5\%} + 3\% \times 1_{E>5\%}.$$

Verification Coupon Formula 2

- We can check that our decomposition is correct, considering the four components in two cases: Libor above or below 5%.
- This is done in the following Table

Component	$E < 5\%$	$E > 5\%$
Fixed	0.5%	0.5%
Floating	E	E
Short AON $K = 5\%$	0	$-E$
Long 0.03 CON $K = 5\%$	0	3%
Sum	$E + 0.5\%$	3.5%

Table 4: Verification of the payoff decomposition. First column: different option components. Second and third columns: payoffs depending on the value of the LIBOR rate.

Questions

Question.

- Consider the following coupon formula

$$\textbf{CPN RATE} = \textbf{LIBOR} + \textbf{10 BP; MAX CPN 6.25\%.}$$

- Plot the payoff function
- Determine a possible decomposition of the above coupon formula.

Answer I.

Another possible decomposition is given below

$$
\begin{aligned}
c &= \min\left(L + 0.1\%, 6.25\%\right) \\
&= \min\left(L + 0.1\% - 6.25\%, 0\right) + 6.25\% \\
&= -\max\left(6.25\% - \left(L + 0.1\%\right), 0\right) + 6 \\
&= -\max\left(6.15\% - L, 0\right) + 6.25\% \\
&= \underbrace{6.25\%}_{\text{(fixed rate)}} \; \underbrace{-\max\left(6.15\% - L, 0\right)}_{\text{(short floorlet)}}.
\end{aligned}
$$

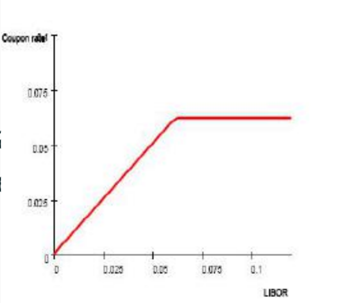


Figure 2: Coupon Payoff

Answer: Verification

Home Questions

Consider the following coupon formulas

- CPN RATE = LIBOR + 10 BP; MAX CPN 6.25%.
- CPN RATE = 14.5% - LIBOR; MAX CPN 6%, MIN CPN = 4%.
- CPN RATE = 16.22%-2 LIBOR if positive; otherwise, 2%.
- CPN RATE = 0.85 LIBOR; MIN CPN 4%.

Assume the Libor is always positive.

Pricing Models for Caps & Floors: Black, Bachelier, Shifted Black

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SMM269 Fixed Income - Academic Year 2019-20

Main References:

Useful Readings

- Brigo Damiano and Fabio Mercurio, Interest Rate Models: Theory and Practice, Springer Finance 2001.
- Pietro Veronesi. Fixed Income Securities. **Chapter 20.**
- Bruce Tuckman, Angel Serrat. Fixed Income Securities: Tools for Today's Markets, 3rd Edition **Chapter 18**.
- Interest rate derivatives in the negative-rate environment Pricing with a shift, Deloitte, Feb 2016.
- Options valuation strained by quantitative easing, Sungard.

Excel Files

- FI_InterestRateOptions.xlsm
- FI_BlackModel&co.xlsm

Outline

Caps & Floors

Caplet

- A caplet is a call option on the LIBOR rate.
- The amount of the payment of every caplet expiring at time T_i will occur at time T_{i+1} and is equal to:

$$\alpha_{T_i, T_{i+1}} \times (L(T_i, T_{i+1}) - L_x)^+ \times N.$$

- Given that at the option expiry we have:

$$F(T_i, T_i, T_{i+1}) = L(T_i, T_{i+1})$$

we can write the caplet payoff in terms of forward rate

$$\alpha_{T_i, T_{i+1}} \times (F(T_i, T_i, T_{i+1}) - L_x)^+ \times N.$$

- Then a caplet is a call option on the simple forward LIBOR rate.
- In order to price the caplet we need a model for the forward LIBOR rate.

Floorlet

- A floorlet is a put option on the LIBOR rate.
- Similarly, to the caplet, we can write the floorlet payoff in terms of the forward rate
$$\alpha_{T_i, T_{i+1}} \times (L_x - F(T_i, T_i, T_{i+1}))^+ \times N.$$
- Then a **floorlet is a put option on the simple forward LIBOR rate.**
- In order to price the floorlet we need a model for the forward LIBOR rate.
- If we are interested in pricing a floor, we need to price all the floorlets contained in the cap.

Cap

- A cap is a portfolio of caplets on the LIBOR rate.

Cap Payoffs		
t		Trading date
T_0		1^{st} reset date
T_1	$\searrow$	$N \times \alpha_{T_0, T_1} \times (L(T_0, T_1) - L_x)^+$
...	$\searrow$	...
T_{i-1}		
T_i	$\searrow$	$N \times \alpha_{T_{i-1}, T_i} \times (L(T_{i-1}, T_i) - L_x)^+$
T_{i+1}	$\searrow$	$N \times \alpha_{T_i, T_{i+1}} \times (L(T_i, T_{i+1}) - L_x)^+$
...	$\searrow$	...
T_n		$N \times \alpha_{T_{n-1}, T_n} \times (L(T_{n-1}, T_n) - L_x)^+$

- At each date a different caplet is expiring.
- Notice that last reset date occurs at time T_{n-1} and T_n is last payment date. Usually, $t < T_0$, i.e. the trading date represents also the first reset date.

Floor

- A floor is a portfolio of floorlet on the LIBOR rate.
- The same considerations as for a cap hold.

Floor Payoffs		
t		Trading date
T_0		1^{st} reset date
T_1	$\searrow$	$N \times \alpha_{T_0, T_1} \times (L_x - L(T_0, T_1))^+$
...	$\searrow$	...
T_{i-1}		
T_i	$\searrow$	$N \times \alpha_{T_{i-1}, T_i} \times (L_x - L(T_{i-1}, T_i))^+$
T_{i+1}	$\searrow$	$N \times \alpha_{T_i, T_{i+1}} \times (L_x - L(T_i, T_{i+1}))^+$
...	$\searrow$	...
T_n		$N \times \alpha_{T_{n-1}, T_n} \times (L_x - L(T_{n-1}, T_n))^+$

At-the-money caps & floors: The put-call parity

- The difference between the payoff of a cap and a floor is

$$\sum_{i=0}^{n-1} \alpha_{T_i, T_{i+1}} \left(L_{i,i+1} - L_x \right),$$

  i.e. the payoff of a forward starting payers swap.
- Therefore, if cap/floor strike L_x is equal to the forward swap rate the swap has zero value and therefore the cap and floor must have the same price and we say that the cap and the floor are at-the-money.
- A caplet and a floorlet written on the same reference rate and having same reset and payment dates are at-the-money if their common strike is equal to the simple forward LIBOR.

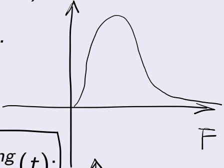
Fact (Caplet Pricing using martingale modelling)

If $P(t, T_{i+1})$ is the observed discount factor for maturity T_{i+1}, the price of the caplet is:

$$\alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times \mathbb{E}_t^{T_{i+1}} \left(F(T_i, T_i, T_{i+1}) - L_x\right)^+ .$$

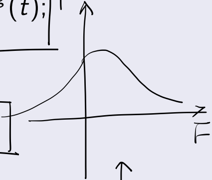
Under the pricing T_{i+1} measure $F(t, T_i, T_{i+1})$ is a martingale.
The most popular models assume that the forward libor is

- *a* **lognormal** *martingale (Black Model)*



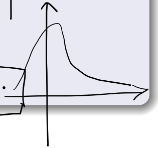
$$dF(t, T_i, T_{i+1}) = \sigma_i F(t, T_i, T_{i+1})\, dW^{pricing}(t);$$

- *or a* **Gaussian** *martingale (Bachelier Model)*



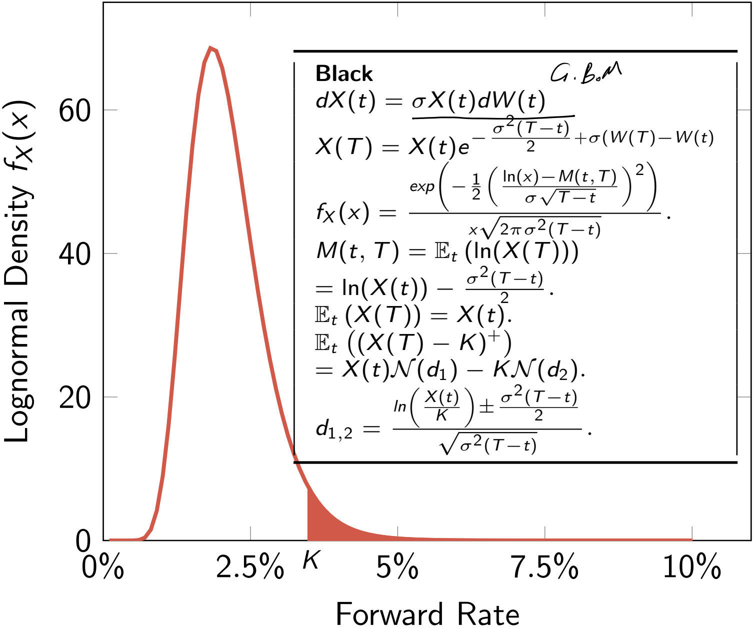
$$dF(t, T_i, T_{i+1}) = \sigma_i dW^{pricing}(t);$$

- *or a* **shifted lognormal** *martingale (Displaced Black)*



$$dF(t, T_i, T_{i+1}) = \sigma_i \left(F(t, T_i, T_{i+1}) + \delta\right) dW^{pricing}(t).$$

Black Model for caplets & floorlets, caps & floors



Black

$$dX(t) = \sigma X(t) dW(t)$$

$$X(T) = X(t) e^{-\frac{\sigma^2 (T-t)}{2} + \sigma (W(T) - W(t))}$$

$$f_X(x) = \frac{exp\left(-\frac{1}{2}\left(\frac{\ln(x) - M(t,T)}{\sigma\sqrt{T-t}}\right)^2\right)}{x\sqrt{2\pi\sigma^2(T-t)}}.$$

$$M(t, T) = \mathbb{E}_t\left(\ln(X(T))\right)$$

$$= \ln(X(t)) - \frac{\sigma^2(T-t)}{2}.$$

$$\mathbb{E}_t\left(X(T)\right) = X(t).$$

$$\mathbb{E}_t\left((X(T) - K)^+\right)$$

$$= X(t)\mathcal{N}(d_1) - K\mathcal{N}(d_2).$$

$$d_{1,2} = \frac{ln\left(\frac{X(t)}{K}\right) \pm \frac{\sigma^2(T-t)}{2}}{\sqrt{\sigma^2(T-t)}}.$$

Fact (Lognormal Martingale Model and the Black Formula)

Assume $X(t)$ to be a lognormal martingale under the "pricing" measure

$$dX(t) = \sigma_X(t) X(t)\, dW^{num}(t),$$

where σ_X refers to the percentage or lognormal volatility. It follows that

$$\mathbb{E}_t\left((X(T) - K)^+ \right) = X(t)\mathcal{N}(d_1) - K\mathcal{N}(d_2),$$

where

$$d_{1,2} = \frac{ln\frac{X(t)}{K} \pm \frac{1}{2}V(t,T)}{\sqrt{V(t,T)}},$$

and

$$V(t,T) = \sigma_X^2(T-t).$$

If $\sigma_X^2(t)$ is time-varying, then $V(t,T) = \int_t^T \sigma_X^2(s)ds$.

The Black Formula for pricing caplets and floorlets

If the simple forward rate has dynamics under the pricing measure given by

$$dF(t, T_i, T_{i+1}) = \sigma_i F(t, T_i, T_{i+1})\, dW^{pricing}(t),$$

the values at date t of European options with strike L_x and maturity date T_i on the simple forward rate $F(T_i, T_{i+1}, T_{i+1})$, are:

$$\begin{aligned} Caplet(t) &= P(t, T_{i+1}) \times \left(F(t, T_i, T_{i+1}) \times \mathcal{N}\left(d_1^i\right) - L_x \times \mathcal{N}\left(d_2^i\right)\right) \alpha_{i,i+1} \\ Floorlet(t) &= P(t, T_{i+1}) \times \left(L_x \times \mathcal{N}\left(-d_2^i\right) - F(t, T_i, T_{i+1}) \times \mathcal{N}\left(-d_1^i\right)\right) \alpha_{i,i+1} \end{aligned}$$

Black-Formula

where:

$$\begin{aligned} d_1^i &= \frac{\ln\left(\frac{F(t,T_i,T_{i+1})}{L_x}\right) + \frac{1}{2}V(t, T_i)}{\sqrt{V(t, T_i)}}, d_2^i = d_1^i - \sqrt{V(t, T_i)}, \\ \mathcal{N}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz, V(t, T_i) = \sigma^2 \times (T_i - t). \end{aligned}$$

The Black Greeks

Let T_i to be the caplet maturity and T_{i+1} to be the payment date.

- The Delta, i.e. the first derivative of the caplet price with respect to the underlying rate $F(t, T_i, T_{i+1})$ (i.e. $(-1) \times DV01$), is given by:

$$\Delta = P(t, T_{i+1}) \times \mathcal{N}\left(d_1^i\right) \times \alpha_{T_i, T_{i+1}}.$$

- The Gamma, i.e. the second derivative, is given by:

$$\Gamma = \frac{P(t, T_{i+1})}{\sqrt{V(t, T_i)} \times F(t, T_i, T_{i+1})} \times n\left(d_1^i\right) \times \alpha_{T_i, T_{i+1}}.$$

  where $n(x)$ is the standard normal density function.

- The Vega, i.e. the first derivative of the caplet price with respect to the volatility parameter is given by:

$$\upsilon = P(t, T_{i+1}) \sqrt{T_i - t} F(t, T_i, T_{i+1}) \, n\left(d_1^i\right) \times \alpha_{T_i, T_{i+1}}.$$

- Expressions for the remaining Greeks can be found in the Martellini et al. book, page 512.

Example (Pricing a 9x12 caplet using Black)

- The 9m discount factor is 0.9631944. The 12m discount factor is 0.9512294.
- The 9x12 simple forward rate is

$$F = \frac{1}{0.25}\left(\frac{0.9632}{0.9951} - 1\right) = 5.0134\%.$$

- In addition, the caplet strike is 4.50%. The (constant) percentage volatility is 10%. Therefore:

$$d_1 = \frac{\ln\left(\frac{5.0134\%}{4.50\%}\right) + \frac{1}{2}0.1^2 \times 0.75}{\sqrt{0.1^2 \times 0.75}} = 1.3321, d_2 = d_1 - \sqrt{0.1^2 \times 0.75} = 1.2455,$$

and

$$\mathcal{N}(d_1) = 0.908593, \mathcal{N}(d_2) = 0.893533.$$

- Therefore

$$c(t) = 0.951229 \times (5.0134\% \times 0.908593 - 4.50\% \times 0.893533) \times 0.25 = 0.001309.$$

- The caplet DV01 is $-0.951229 \times 0.908593 \times 0.25$.

Pricing a Cap (Floor) using the Black formula

- A cap (floor) is a collection of caplets (floorlets), so it is a basket of call (put) options and its price will depend on the strip of future simple forward rates ($t < T_0 < ... < T_{n-1} < T_n$):

$$F(t, T_0, T_1), \cdots, F(t, T_{n-1}, T_n),$$

  characterized by percentage volatilities (named `forward-forward volatilities`):

$$\sigma_0, \cdots, \sigma_{n-1}.$$

- By additivity, the cap price is equal to the sum of the prices of the single caplets[a]:

$$\sum_{i=0}^{n-1} \alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times \left(F(t, T_i, T_{i+1}) \times \mathcal{N}\left(d_1^i\right) - L_x \times \mathcal{N}\left(d_2^i\right)\right) \times N.$$

- Similarly for the floor, we have

$$\sum_{i=0}^{n-1} \alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times \left(L_x \times \mathcal{N}\left(-d_2^i\right) - F(t, T_i, T_{i+1}) \times \mathcal{N}\left(-d_1^i\right)\right) \times N.$$

[a] In a standard cap the caplet that resets in t is not included. So $T_0 > t$

Example (Pricing a Cap using a flat volatility)

Sheet: ExampleCapFloorATM

Time	Discount Factor	Period	Fwd. Rate	d_1	d_2	Fwd Prem	Black	Caplet
0	100.00%							
0.5	99.501%	0 x 0.5						
1	98.807%	0.5 x 1	1.405%	-0.8031	-0.9091	0.017%	0.0167%	**84**
1.5	98.069%	1 x 1.5	1.506%	-0.0687	-0.2187	0.076%	0.0741%	**370**
2	97.239%	1.5 x 2	1.707%	0.6585	0.4748	0.222%	0.2155%	**1,078**
							Cap	**1,532**

Table: Cap Term: 2 years, Caplet Tenor: 0.5; Strike: ATM; Flat Vol. 15%, Face Value 1,000,000

The Cap price is the sum of caplet prices 6x12, 12x18 and 18x24:

$$84 + 370 + 1,078 = 1,532.$$

Let us detail the computation concerning the 18x24 caplet.

Example ((...continued: pricing the 18x24 caplet))

1. We compute the ATM strike

$$K = \frac{0.99501 - 0.97239}{0.5\,(98.807\% + 98.069\% + 97.239\%))} = 1.5385\%.$$

2. We compute d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{1.707\%}{1.5385\%}\right) + \frac{1}{2} \times (0.15)^2 \times 1.5}{0.15 \times \sqrt{1.5}} = 0.6585;\, d_2 = 0.6585 - 0.15 \times \sqrt{1.5}$$

3. We compute the forward premium

$$(1.707\% \times \mathcal{N}\,(0.6585) - 1.5385\% \times \mathcal{N}(0.4748)) \times (2 - 1.5) = 0.111\%.$$

4. The present value of the forward premium is

$$0.97239 \times 0.111\% = 0.1078\%.$$

5. The 18x24 caplet price is

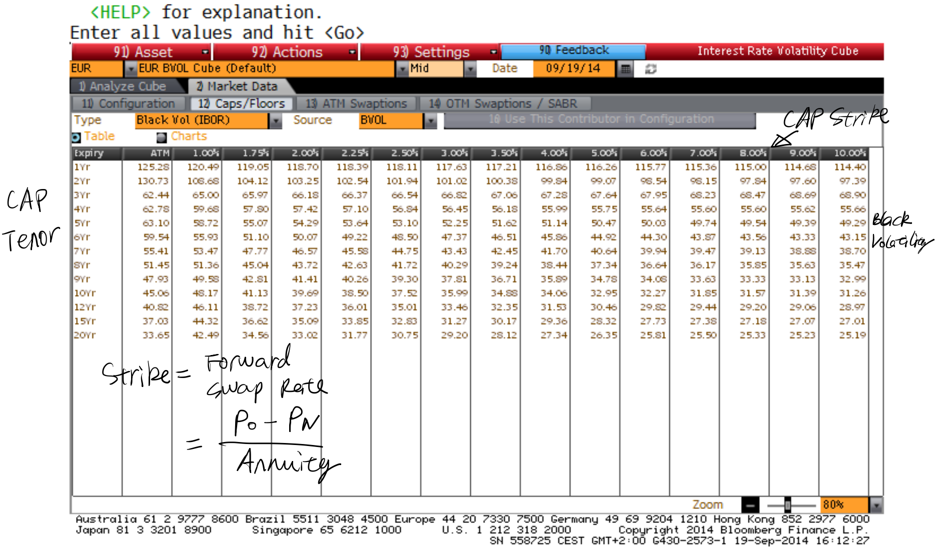
$$1,000,000 \times 0.1078\% = 1,078.$$

Market Quotes for caps
The volatility surface

The term structure of implied volatilities

See Veronesi, section 20.1.1

- The OTC market quotes the implied volatility for caps and floors (i.e. the strike L_x is set equal to the forward swap rate) of different maturities and different strikes.
- Then the implied volatilities are transformed in prices, pricing all the caplets using the Black formula (or some variants of it) and the same volatility.
- The quoted implied volatility is called `flat or par volatility`.
- The flat volatility is a kind of "average volatility" of the set of individual caplet volatilities: in a 2 year US cap it would apply to all seven caplets.
- If we plot these implied volatilities against maturity, we obtain the so called term structure of implied cap volatilities.



<HELP> for explanation.
Enter all values and hit <Go>

91) Asset | 92) Actions | 93) Settings | 90 Feedback | Interest Rate Volatility Cube

EUR | EUR BVOL Cube (Default) | Mid | Date 09/19/14

1) Analyze Cube | 2) Market Data

11) Configuration | 12) Caps/Floors | 13) ATM Swaptions | 14) OTM Swaptions / SABR

Type: Black Vol (IBOR) | Source: BVOL | 16) Use This Contributor in Configuration

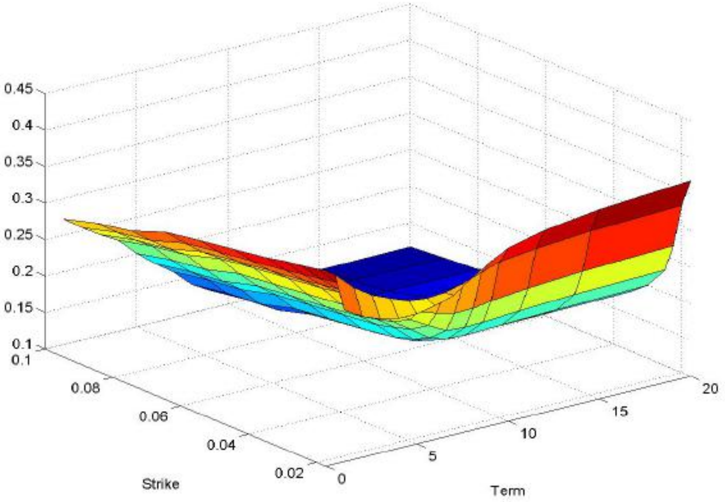
Table | Charts

Expiry	ATM	1.00%	1.75%	2.00%	2.25%	2.50%	3.00%	3.50%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%
1Yr	125.28	120.49	119.05	118.70	118.39	118.11	117.63	117.21	116.86	116.26	115.77	115.36	115.00	114.68	114.40
2Yr	130.73	108.68	104.12	103.25	102.54	101.94	101.02	100.38	99.84	99.07	98.54	98.15	97.84	97.60	97.39
3Yr	62.44	65.00	65.97	66.18	66.37	66.54	66.82	67.06	67.28	67.64	67.95	68.23	68.47	68.69	68.90
4Yr	62.78	59.68	57.80	57.42	57.10	56.84	56.45	56.18	55.99	55.75	55.64	55.60	55.60	55.62	55.66
5Yr	63.10	58.72	55.07	54.29	53.64	53.10	52.25	51.62	51.14	50.47	50.03	49.74	49.54	49.39	49.29
6Yr	59.54	55.93	51.10	50.07	49.22	48.50	47.37	46.51	45.86	44.92	44.30	43.87	43.56	43.33	43.15
7Yr	55.41	53.47	47.77	46.57	45.58	44.75	43.43	42.45	41.70	40.64	39.94	39.47	39.13	38.88	38.70
8Yr	51.45	51.36	45.04	43.72	42.63	41.72	40.29	39.24	38.44	37.34	36.64	36.17	35.85	35.63	35.47
9Yr	47.93	49.58	42.81	41.41	40.26	39.30	37.81	36.71	35.89	34.78	34.08	33.63	33.33	33.13	32.99
10Yr	45.06	48.17	41.13	39.69	38.50	37.52	35.99	34.88	34.06	32.95	32.27	31.85	31.57	31.39	31.26
12Yr	40.82	46.11	38.72	37.23	36.01	35.01	33.46	32.35	31.53	30.46	29.82	29.44	29.20	29.06	28.97
15Yr	37.03	44.32	36.62	35.09	33.85	32.83	31.27	30.17	29.36	28.32	27.73	27.38	27.18	27.07	27.01
20Yr	33.65	42.49	34.56	33.02	31.77	30.75	29.20	28.12	27.34	26.35	25.81	25.50	25.33	25.23	25.19

Zoom 80%

Figure: The volatility surface of cap volatilities.

The implied cap volatility surface



The volatility surface of cap volatilities. June 2005

1. How to use the volatility surface

- Let us suppose we aim to price a 3 year cap with 2% strike written on EUR LIBOR
  - This cap is made of 6x12, 12x18, 18x24, 24x30, and 30x36 caplets.
  - From the volatility surface, we notice that the 3 year flat volatility corresponding to a strike of 2% is equal to 66.18%.
- Therefore, we price all the above caplets using the Black formula in which we input the volatility value of 66.18%.

2. How to use the volatility surface

- Let us suppose we aim to price a 4 year cap with 2% strike written on EUR LIBOR
  - This cap is made of 6x12, 12x18, 18x24, 24x30, 30x36, 36x42 and 42x48 caplets.
  - From the Volatility surface, we notice that the 4 year flat volatility corresponding to a strike of 2% is equal to 57.80%.
- Therefore, we price all the above caplets using the Black formula in which we input the volatility value of 57.80%.
- Notice that the 3 and 4-years caps have in common five caplets (6x12, 12x18, 18x24, 24x30, 30x36).
- These caplets are priced with a different volatility depending on the cap maturity.
- It is like when we price two bonds with different maturity using the yield-to-maturity (ytm). Cash flows falling on the same dates are discounted using different ytm depending to the reference bond.
- It is a market convention. It is important to know that it is used.

Flat Volatility

- When we price a cap we assume that the same volatility applies to all of the caplets making up the cap.
- Indeed, this is the market convention to quote caps and floors. The quoted volatility is called flat volatility
  Flat Volatility. The flat volatility of a **cap** with maturity T is the quoted volatility of $\sigma(T)$ that must be inserted in the Black (or Bachelier/Displaced Black) formula for each and every caplet that makes up the cap, in order to obtain a dollar price for the cap.
- Notice that the flat volatility is applied to caps, not caplets, similarly to the concept of yield to maturity that is applied to bonds and not zero-coupon bond.

Example (Using the volatility surface)

Given the market information about the volatility surface, price ATM caps with 1, 2 and 3 years to expiry. Caplerts have semi-annual tenor. The following information about the discount curve is available.

Term	Discount Factor
0	100.000%
0.5	99.501%
1	98.807%
1.5	98.069%
2	97.239%

Example ((...ctd) Computing ATM Strikes)

See Excel File: FI_BlackModel&co sheet: ExampleCapVolSurface

For ATM Caps, the strike is equal to the forward swap rate

The strike of the 1 Y ATM cap is

$$\frac{0.99501 - 0.98897}{0.5 \times (0.98807)} = 1.4948\%$$

The strike of the 2 Y ATM cap is

$$\frac{0.99501 - 0.97239}{0.5 \times (0.98807 + 0.98069 + 0.9239)} = 1.5382\%$$

The strike of the 3 Y ATM cap is

$$\frac{0.99501 - 0.95300}{0.5 \times (0.98807 + 0.98069 + 0.9239 + 0.9620 + 0.9530)} = 1.7302\%$$

Example ((...ctd) Pricing the caps)

Table: Pricing ATM caps (Strike=forward swap rate) with different tenors and using Flat Volatility. Caplets tenor: 6m; Face Value 1,000,000.

1 Year Cap

Time	DF	StartxEnd	Fwd Rate	Strike	Fwd Vol	d_1	d_2	Caplet
0	100.000%							
0.5	99.501%	0 x 0.5						
1	98.807%	0.5 x 1	1.405%	1.405%	125.280%	0.442932	-0.44293	2375
ATM Strike		1.4048%					**Cap**	2,375

2 Years Cap

Time	DF	StartxEnd	Fwd Rate	Strike	Fwd Vol	d1	d2	Caplet
0	100.000%							
0.5	99.501%	0 x 0.5						
1	98.807%	0.5 x 1	1.405%	1.538%	130.730%	0.36405	-0.56035	2270
1.5	98.069%	1 x 1.5	1.505%	1.538%	130.730%	0.637004	-0.6703	3550
2	97.239%	1.5 x 2	1.707%	1.538%	130.730%	0.865647	-0.73546	4968
ATM Strike		1.5382%					**Cap**	**10,788**

3 Years Cap

Time	DF	StartxEnd	Fwd Rate	Strike	Fwd Vol	d1	d2	Caplet
0	100.000%							
0.5	99.501%	0 x 0.5						
1	98.807%	0.5 x 1	1.405%	1.730%	62.440%	-0.25116	-0.69267	694
1.5	98.069%	1 x 1.5	1.505%	1.730%	62.440%	0.088963	-0.53544	1439
2	97.239%	1.5 x 2	1.613%	1.730%	62.440%	0.199412	-0.42499	1718
2.5	96.200%	2 x 2.5	1.807%	1.730%	62.440%	0.38147	-0.24293	2274
3	95.300%	2.5 x 3	1.889%	1.730%	62.440%	0.582466	-0.4048	3653
ATM Strike		1.7302%					**Cap**	**9,777**

Example ((...ctd): pricing the 18x24 caplet in the 2 year cap))

Let us detail the computation concerning the 18x24 caplet.

1. We compute the ATM strike

$$K = \frac{0.99501 - 0.97239}{0.5\,(98.807\% + 98.069\% + 97.239\%))} = 1.5382\%.$$

2. We compute d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{1.707\%}{1.5385\%}\right) + \frac{1}{2} \times (1.30730)^2 \times 1.5}{1.30730 \times \sqrt{1.5}} = 0.86565$$

   and then $d_2 = 0.86565 - 1.30730 \times \sqrt{1.5} = -0.73546.$

3. We compute the forward premium

$$(1.707\% \times \mathcal{N}(0.86565) - 1.5385\% \times \mathcal{N}(-0.73546)) \times (2 - 1.5) = 0.510855\%.$$

4. The present value of the forward premium is

$$0.97239 \times 0.510855\% = 0.496751\%.$$

5. The 18x24 caplet price is $1,000,000 \times 0.496751\% = 4,968.$

Example (At the money floor)

- In our example, relative to pricing ATM caps we have set the strike equal to the forward swap rate
- In particular, for the 2 year cap we set $K = 1.5382\%$, (1st reset in 6m, last payment in 24 m, payment dates: 12m, 18m, 24m).
- The we have obtained that the cap price is 10,788;
- The corresponding ATM floor is also priced at 10,788.

c. so that the value of the forward starting swap with semi-annual payments and 18m tenor is 0. See indeed the remark at page 9.

- The same holds for the 1 and 3 year caps and floors.

Forward Volatility I

Source Veronesi, page. 688-689

- Let us consider the market quotations of page 19.
  - If we aim to price a 2 year cap with strike 2%, we have to price the caplets 6x12, 12x18, 18x24 using a flat implied volatility of 103.25%.
  - If we aim to price a 3 year cap with strike 2%, we have to price the caplets 6x12, 12x18, 18,24, 24x30 and 30x36 using a flat implied volatility of 66.18%.
- This implies that the caplets 6x12, 12x18 and 18x24 have a volatility of 103.25% when they are part of the 2-years cap, but a lower volatility (66.18%) when part of a 3-year cap.
- The fact that the same caplet has different volatilities depending on which cap it is part of may suggest at first that there is a large inconsistency in the traders' quotes of caps, but this is in fact not correct.

Forward Volatility II

- The **quoted (or flat) volatility**, which is a convention that traders in the market place adopt to exchange caps and floors, and a second one is the no **arbitrage (forward) volatility**, which instead would call for the same caplet to have the same volatility independent of which cap it is part of. The forward volatility of a **caplet** with maturity T and strike rate K is the volatility $\sigma_F(T)$ that characterizes that particular caplet, independent of which cap the caplet belongs to
- Forward volatility is applied to caplets while the flat volatility is applied to caps.
- The forward volatilities are going to be the no arbitrage ingredients for the pricing of other more complex securities.
- Flat volatilities are a nonlinear average of forward volatilities, so in general their appear to be smoother than forward volatilities

Pricing Using Flat Volatilities

Cap Maturity	Caplet Expiry 1x2	2x3	3x4	4x5
2	v(2)			
3	v(3)	v(3)		
4	v(4)	v(4)	v(4)	
5	v(5)	v(5)	v(5)	v(5)

Pricing using Fwd-Fwd Volatilities

Cap Maturity	Caplet Expiry 1x2	2x3	3x4	4x5
2	σ(1,2)			
3	σ(1,2)	σ(2,3)		
4	σ(1,2)	σ(2,3)	σ(3,4)	
5	σ(1,2)	σ(2,3)	σ(3,4)	σ(4,5)

Figure: Flat vs Forward Volatilities

How do we extract the forward volatilities from flat volatilities?

- For example, let us price a 3 year cap (assume caplets have annual tenor). We can price it in two ways

1. **Using Flat Volatility**

$$Cap(3, v(3)) = caplet(1x2, v(3)) + caplet(2x3, v(3))$$

2. **Using Forward Volatility**

$$Cap(3, \sigma_F(1), \sigma_F(2))) = caplet(1x2, \sigma_F(1)) + caplet(2x3, \sigma_F(2))$$

3. Extracting the Forward Volatility: Given $v(3)$ and $\sigma_F(1)$, we can solve for $\sigma_F(2)$ the equation

$$caplet(1x2, v(3)) + caplet(2x3, v(3)) = caplet(1x2, \sigma_F(1)) + caplet(2x3, \sigma_F(2))$$

4. Some interpolation is however needed.

Extracting Forward Volatilities from Flat Volatilities I

Excel File: FI_BlackMo&Co.xlsx
Sheet: ExampleCapFWDVol

Example (Extracting Forward Volatilities: 1Y FWD VOL)

Table: We set the 1Y FWD VOL equal to the 1Y FLAT VOL

1 Year Cap

Time	DF	StartxEnd	Fwd Rate	Strike	Flat Vol	d1	d2	Caplet	FLAT VOL	FLAT PRICE
0	100.000%									
0.5	99.501%	0 × 0.5								
1	98.807%	0.5 × 1	1.405%	1.405%	118.700%	0.419668	-0.41967	2257	118.70%	2257.4
Tenor		0.5								
Strike		2.0000%					**Cap**	2,257		2257.4
Notional		1000000								

Extracting Forward Volatilities from Flat Volatilities II

Example ((.ctd): Extracting the 2Y FWD VOL)

Table: We set the 1x1.5 and 1.5x2 FWD VOL such that the price of the 2Y CAP is equal to 7,054 (obtained using a flat vol of 103.25%)

2 Years Cap										
Time	DF	StartxEnd	Fwd Rate	Strike	Flat Vol	d1	d2	Caplet	FLAT VOL	FLAT PRICE
0	100.000%									
0.5	99.501%	0 x 0.5								
1	98.807%	0.5 x 1	1.405%	2.000%	118.700%	-0.00124	-0.84057	1488	103.25%	1185.8
1.5	98.069%	1 x 1.5	1.505%	2.000%	98.584%	0.204527	-0.78132	2157	103.25%	2290.9
2	97.239%	1.5 x 2	1.707%	2.000%	98.584%	0.47257	-0.73484	3410	103.25%	3577.9
Tenor		0.5								
Strike		2.0000%					**Cap**	**7,054**		7054.5
Notional		1000000								

Extracting Forward Volatilities from Flat Volatilities III

Example ((.ctd): Extracting the 3Y FWD VOL)

Table: We set the 2x2.5 and 2.5x3 FWD VOL such that the price of the 3Y CAP is equal to 11,318 (obtained using a flat vol of 66.18%)

Time	DF	StartxEnd	Fwd Rate	Strike	Flat Vol	d1	d2	Caplet	FLAT VOL	FLAT PRICE
0	100.000%									
0.5	99.501%	0 x 0.5								
1	98.807%	0.5 x 1	1.405%	2.000%	118.700%	-0.00124	-0.84057	1488	66.18%	496.1
1.5	98.069%	1 x 1.5	1.505%	2.000%	98.584%	0.204527	-0.78132	2157	66.18%	1208.3
2	97.239%	1.5 x 2	1.613%	2.000%	98.584%	0.274482	-0.71136	2449	66.18%	2175.2
2.5	96.200%	2 x 2.5	1.807%	2.000%	64.423%	0.164298	-0.47993	1876	66.18%	4000.0
3	95.300%	2.5 x 3	1.889%	2.000%	64.423%	0.453136	-0.56548	3349	66.18%	3438.3
Tenor		0.5								
Strike		2.0000%					**Cap**	**11,318**		11,318
Notional		1000000								

Handwritten annotations:

FWD Vol (written under the "Flat Vol" header)

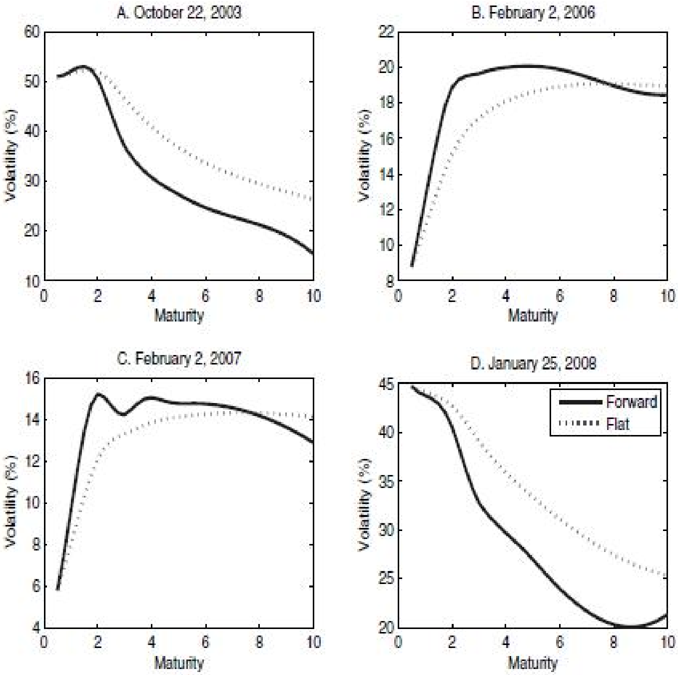
FWD Vol are assumed to be piece-wise constant

Extracting Forward Volatilities from Flat Volatilities IV

Example

Table: The term structure of flat and forward volatilities

TERM	FLAT VOL	FWD VOL
1	118.70%	118.700%
2	103.25%	98.584%
3	66.18%	64.423%


A. October 22, 2003
B. February 2, 2006
C. February 2, 2007
D. January 25, 2008
Volatility (%)
Maturity
Forward
Flat
60
50
40
30
20
10
22
20
18
16
14
12
10
8
16
14
12
10
8
6
4
45
40
35
30
25
20
0
2
4
6
8
10

Coping with Negative Rates Bachelier and Displaced Black models

The past

- Rates are significantly positive.
- Volatilities are at normal levels.
- Quotes are in log-normal volatility or premium.
- The Black model assumes that the underlying has a zero probability of becoming negative.

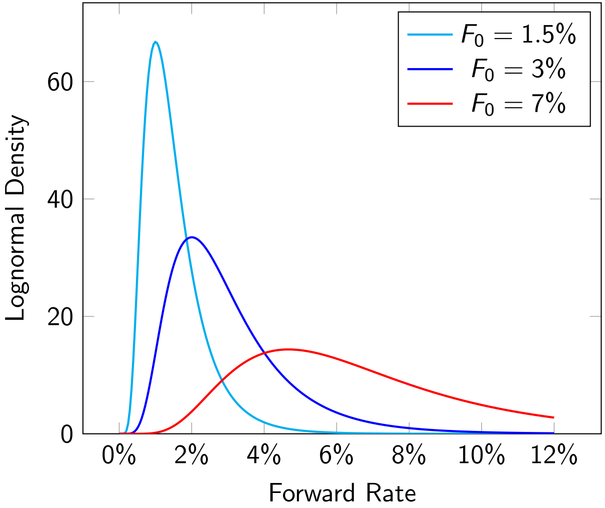


Figure: Probability density of Black's model for current forward rates $F \in 1.5\%; 3\%; 7\%$, volatility $\sigma = 0.3$ and time $T = 3$. Source:

Table: ICE LIBOR Rates on November 30th, 2016

Tenor	CHF	EUR	JPY	GBP	USD
Spot Next	-0.7818	-0.41	-0.08314	0.225	0.43056
1 Week	-0.7952	-0.39386	-0.07329	0.25119	0.45822
1 Month	-0.8178	-0.37957	-0.10971	0.26075	0.62367
2 Month	-0.7856	-0.34643	-0.11086	0.32438	0.7425
3 Month	-0.749	-0.325	-0.06743	0.38463	0.93417
6 Month	-0.68	-0.21871	0.01114	0.55038	1.28878
1 Year	-0.5122	-0.07971	0.10529	0.79244	1.639

Table: ICE Swap Rates on November 30th, 2016

Tenor	EUR Rates 1200	GBP Rates 1100	USD Rates 1100
1 Year	-0.299	0.429	1.103
2 Years	-0.17	0.644	1.318
3 Years	-0.114	0.736	1.514
4 Years	-0.037	0.832	1.675
5 Years	0.061	0.926	1.806
6 Years	0.172	1.018	1.917
7 Years	0.29	1.106	2.01
8 Years	0.406	1.189	2.086
9 Years	0.517	1.265	2.15
10 Years	0.618	1.332	2.204
12 Years	0.785	1.436	
15 Years	0.959	1.528	2.372
20 Years	1.098	1.573	2.447
25 Years	1.136	1.558	
30 Years	1.15	1.536	2.489

Which model with negative rates?

- We have to handle curves

  Generally the negative rates do not have that impact on curves construction and is a standard process these days

  The appearance of significant basis spreads have much more impact
- We still need to price options

  Caps/Floors

  Swaptions

  CMS Caps/Floors

  CMS Spread Options

  even FRN: coupons have an implied floor at 0.
- Market quotes for volatilities of negative strikes do not always exist. We have to extrapolate market-quoted volatilities into the negative domain.

Expiry	ATM	0.50%	1.00%	1.50%	2.00%	2.25%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%	6.00%	7.00%	10.00%
1Yr				553.88	460.67	434.53	414.60	909.88	735.93	659.30	613.79	582.57	541.22	514.28	468.53
18Mo								666.98	525.62	465.69	429.45	404.24	370.37	348.00	309.34
2Yr													578.43	468.66	369.27
3Yr															702.01
5Yr				128.18	196.22	166.42	150.63	132.80	122.59	115.86	111.07	107.47	102.42	99.04	93.40
6Yr			132.36	148.04	112.79	104.15	97.88	89.35	83.79	79.89	77.01	74.81	71.68	69.58	66.17
7Yr			196.47	108.50	88.17	82.34	77.94	71.75	67.63	64.70	62.54	60.88	58.52	56.96	54.43
8Yr			138.45	91.20	74.84	69.88	66.07	60.62	56.93	54.31	52.36	50.87	48.77	47.39	45.25
9Yr		193.23	116.93	81.06	66.97	62.58	59.18	54.28	50.94	48.56	46.79	45.43	43.51	42.25	40.28
10Yr		148.67	104.90	74.60	61.95	57.96	54.85	50.35	47.28	45.09	43.46	42.21	40.45	39.30	37.50
12Yr	104.02	122.26	92.57	67.11	55.82	52.16	49.27	45.01	42.03	39.86	38.22	36.95	35.13	33.93	32.08
15Yr	125.06	112.35	86.18	62.71	52.18	48.74	46.01	41.94	39.07	36.95	35.34	34.07	32.23	30.99	28.99
20Yr	105.36	125.12	88.27	62.29	51.45	47.96	45.19	41.06	38.14	35.97	34.29	32.96	31.00	29.63	27.29
25Yr	131.08		103.12	65.89	53.40	49.53	46.51	42.07	38.96	36.66	34.89	33.49	31.42	29.98	27.49
30Yr	79.02		195.08	71.67	56.28	51.84	48.44	43.53	40.15	37.67	35.78	34.29	32.10	30.57	27.92

Figure: Black Volatilities quotations nowadays (Nov. 2016).

Negative rates are a challenge for several reasons

- It is a particular challenge for structured products with embedded interest rate options.
- The Black model for valuing options assumes strikes cannot go negative, which is a problem for interest rate options books.
- Indeed the most widely used model for valuing the derivatives assumes rates cannot go below zero.
- Changing the models is a huge undertaking and could result in losses when positions with 0% floors are revalued.

Implications for a banks business

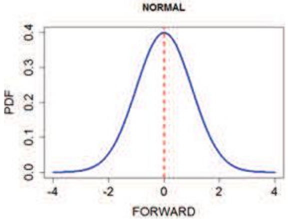
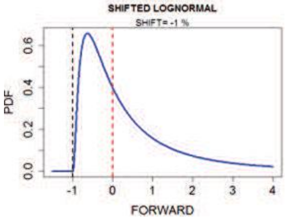
- It might simply refuse to accept negative forward rates, or if there is a process for identifying and alerting exceptions - that is, items that are not processed successfully - it might tell the middle office to check that its inputs and curves are correct.
- But given the Black model cannot handle negative rates, an overhaul is needed to allow for accurate valuations.
- Zero Strike Floors (Implicit in many bonds)
- Options with negative strikes
- Model Choice (Construction of Volatility Surfaces/Hedges/Exotics/...)
- IT Systems (Implementation of new models, adjusting existing models): If a model does crash, the impact for the valuation team will depend on the quality of the code in the system.

Figure: An example of a corporate FRN bond with an implicit option with strike at 0

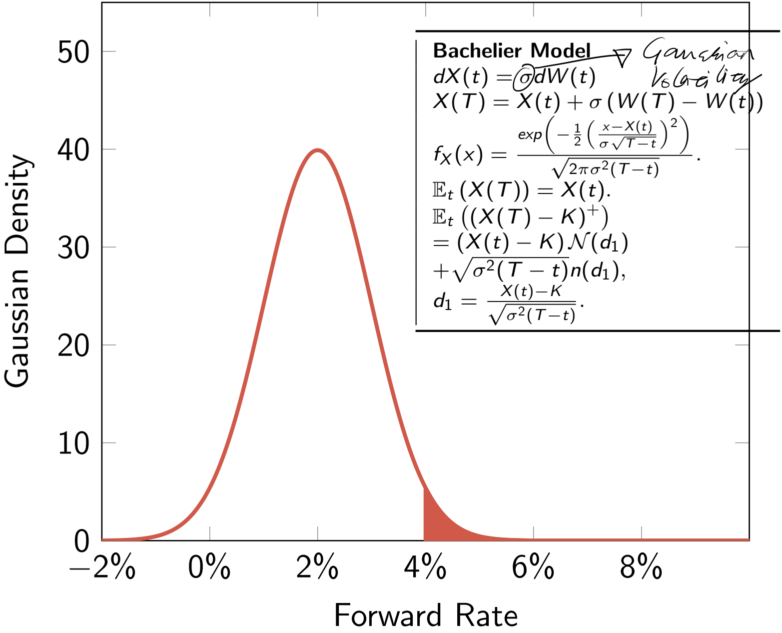
New Interest rate modelling

If a bank is to change the way it prices implied volatility, it has three real alternatives.

- One option is to move to a normally distributed model that allows for positive and negative strikes (Bachelier or Hull and White, a variant of the Vasicek model).
- Others have looked to retain the Black model but shift the strikes into the positive.
- Some have also looked at using prices from the interdealer broker market instead of modelling it themselves. Brokers do not quote all maturities, however, so the bank would have to interpolate prices for the gaps.
- Interpolation in prices is very dangerous because these are highly non-linear prices. The risk is to end up with a gross approximation and inaccuracy.

Model	Probability density	Advantages	Disadvantages
Hull-White, Bachelier		Closed-form formula exists. Model can be readily used.	Extreme negative values are possible (with small probability)
Shifted Black		Closed-form formula exists. Negative values below the shift are not possible.	Requires as input an appropriate volatility that may not be quoted in the market.

The Bachelier Model



Bachelier Model

$$dX(t) = \sigma dW(t)$$

$$X(T) = X(t) + \sigma\left(W(T) - W(t)\right)$$

$$f_X(x) = \frac{exp\left(-\frac{1}{2}\left(\frac{x - X(t)}{\sigma\sqrt{T-t}}\right)^2\right)}{\sqrt{2\pi\sigma^2(T-t)}}.$$

$$\mathbb{E}_t\left(X(T)\right) = X(t).$$

$$\mathbb{E}_t\left((X(T) - K)^+\right)$$

$$= (X(t) - K)\mathcal{N}(d_1)$$

$$+\sqrt{\sigma^2(T-t)}n(d_1),$$

$$d_1 = \frac{X(t) - K}{\sqrt{\sigma^2(T-t)}}.$$

Fact (Gaussian Martingale Model and the Bachelier formula)

Assume $X(t)$ to be a Gaussian martingale under the "pricing" measure

$$dX(t) = \sigma_X(t)\, dW^{num}(t).$$

where σ_X refers to the absolute or Gaussian volatility. It follows that

$$\mathbb{E}_t\left(\left(X(T) - K\right)^+\right) = (X(t) - K)\mathcal{N}(d_1) + \sqrt{V(t,T)}n(d_1),$$

where

$$d_1 = \frac{X(t) - K}{\sqrt{V(t,T)}},$$

and

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

$$V(t,T) = \sigma_X^2(t).$$

If $\sigma_X(t)$ is time varying, then $V(t,T) = \int_t^T \sigma_X^2(s)ds$.

The Bachelier Formula for pricing caplets and floorlets

If the simple forward rate has dynamics given by

$$dF(t, T_i, T_{i+1}) = \sigma_i dW^{pricing}(t),$$

the values at date t of European options with strike L_x and maturity date T_i on the simple forward rate $F(T_i, T_{i+1}, T_{i+1})$, are:

$$Caplet(t) =$$
$$P(t, T_{i+1}) \times \left((F(t, T_i, T_{i+1}) - L_x) \mathcal{N}(d_1^i) + \sqrt{V(t, T_i)} n(d_1^i) \right) \alpha_{i,i+1},$$
$$Floorlet(t) =$$
$$P(t, T_{i+1}) \times \left((L_x - F(t, T_i, T_{i+1})) \mathcal{N}(-d_1^i) + \sqrt{V(t, T_i)} n(d_1^i) \right) \alpha_{i,i+1},$$

where:

$$d_1^i = \frac{F(t, T_i, T_{i+1}) - L_x}{\sqrt{V(t, T_i)}}, \ \mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz, n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Example (Pricing a 9x12 caplet using Bachelier formula)

- The 9m discount factor is 1.0010 (9m LIBOR is -0.131%). The 12m discount factor is 1.0015 (12m LIBOR is -0.203%).
- The caplet strike is -**0.2%**. The (constant) absolute volatility is 2.40%.
- The 9x12 simple forward rate is

$$F = \frac{1}{0.25}\left(\frac{1.0010}{1.0015} - 1\right) = -0.2162\%.$$

- In addition

$$d_1 = \frac{-0.2162\% - (-0.2\%)}{\sqrt{(2.40\%)^2 \times 0.75}} = -0.00780,$$

and

$$\mathcal{N}(d_1) = 0.496888, n(d_1) = 0.398942.$$

- Therefore the caplet price is

$$1.0015 \times \left((-0.2162\% - (-0.2\%)) \times 0.496888 + 2.40\% \times \sqrt{0.75} \times 0.398942\right) \times 0.25 = 0.002056.$$

The normal volatility surface

92) Actions | 93) Settings | Interest Rate Volatility Cube

EUR | EUR BVOL Cube (Default) | Mid | Date 03/02/18

95) Analyze Cube | 96) Market Data

11) Configuration | 12) Caps/Floors | 13) ATM Swaptions | 14) OTM Swaptions / SABR

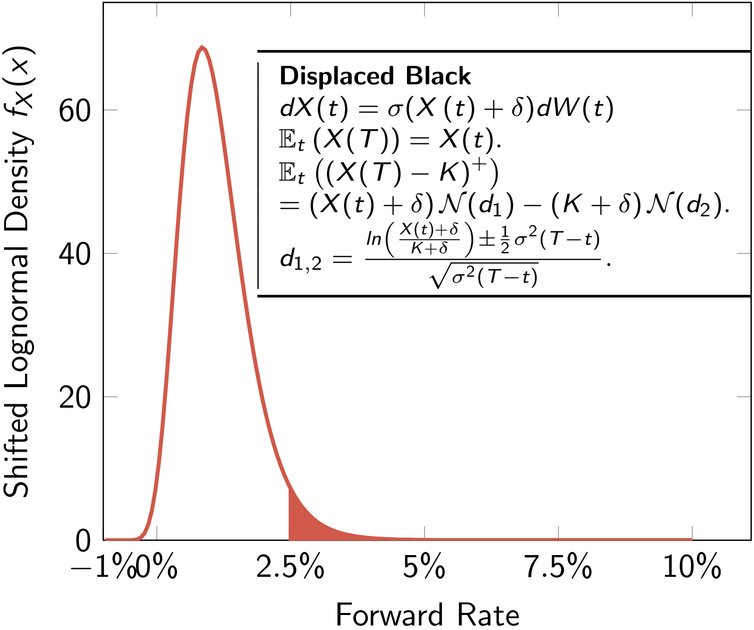
Type Normal Vol (OIS) | Source BVOL | 16) Use This Contributor in Configuration

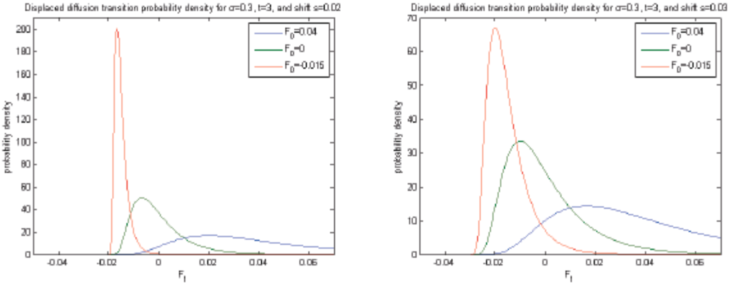
Table | Charts

Expiry	-0.50%	-0.25%	-0.13%	ATM	0.00%	0.13%	0.25%	0.50%	1.00%	1.50%	2.00%	2.50%
1Yr	15.50	11.15	14.14	10.84	17.93	21.72	25.42	32.53	45.90	58.55	70.70	82.50
18Mo	21.37	16.68	19.70	16.78	22.50	25.44	28.53	34.83	47.19	59.03	70.44	81.53
2Yr	24.00	19.54	22.20	21.49	24.17	26.07	28.27	33.53	45.24	57.00	68.53	79.85
3Yr	32.80	27.40	29.06	33.92	31.48	33.69	35.88	40.26	49.10	58.60	68.43	78.32
4Yr	38.12	33.72	35.10	43.32	37.36	39.54	41.68	45.80	53.69	61.92	70.58	79.49
5Yr	41.87	38.46	39.63	49.85	41.63	43.65	45.64	49.47	56.66	63.92	71.41	79.09
6Yr	45.38	42.59	43.51	54.51	45.21	46.97	48.75	52.17	58.53	64.76	71.08	77.53
7Yr	47.66	45.39	46.16	57.53	47.63	49.19	50.78	53.86	59.60	65.22	70.91	76.76
8Yr	50.06	48.07	48.66	59.77	49.88	51.23	52.61	55.33	60.41	65.33	70.27	75.33
9Yr	51.82	50.08	50.56	61.34	51.60	52.77	53.99	56.42	60.99	65.41	69.86	74.45
10Yr	53.06	51.55	51.96	62.54	52.87	53.92	55.02	57.23	61.42	65.48	69.60	73.86
12Yr	54.75	53.59	53.89	63.66	54.61	55.45	56.35	58.18	61.68	65.08	68.50	72.03
15Yr	55.88	55.13	55.36	64.06	55.91	56.56	57.26	58.72	61.55	64.33	67.16	70.09
20Yr	55.36	55.11	55.35	62.85	55.79	56.30	56.85	57.99	60.24	62.49	64.77	67.16
25Yr	54.50	54.55	54.81	61.51	55.21	55.65	56.13	57.11	59.08	61.07	63.11	65.26
30Yr	53.88	54.09	54.36	60.48	54.73	55.13	55.56	56.45	58.25	60.08	61.99	64.00

Figure: Flat volatilities for pricing caps and floors with different terms and strikes in the Bachelier model

Negative Rates: Displaced Black


Displaced Black
$dX(t) = \sigma(X(t) + \delta)dW(t)$
$\mathbb{E}_t (X(T)) = X(t).$
$\mathbb{E}_t ((X(T) - K)^+)$
$= (X(t) + \delta)\mathcal{N}(d_1) - (K + \delta)\mathcal{N}(d_2).$
$d_{1,2} = \frac{ln\left(\frac{X(t)+\delta}{K+\delta}\right) \pm \frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}}.$
Shifted Lognormal Density $f_X(x)$
60
40
20
0
−1%
0%
2.5%
5%
7.5%
10%
Forward Rate



(a) shift $s = 0.02$ (b) shift $s = 0.03$

Figure: Figure (a) and (b) show the transition probability densities of the displaced diffusion model for several rates F and two different shift parameters.

Fact (Displaced Lognormal Martingale Model and the Shifted Black)

Assume $X(t)$ to be a displaced lognormal martingale under the "pricing" measure

$$dX(t) = \sigma_X(t)(X(t) + \delta)dW^{num}(t),$$

where σ_X refers to the displaced lognormal volatility and δ to the displacement parameter. It follows that

$$\mathbb{E}_t\left((X(T) - K)^+\right) = (X(t) + \delta)\mathcal{N}(d_1) - (K + \delta)\mathcal{N}(d_2),$$

where

$$d_{1,2} = \frac{ln\left(\frac{X(t)+\delta}{K+\delta}\right) \pm \frac{1}{2}V(t,T)}{\sqrt{V(t,T)}},$$

and

$$V(t,T) = \sigma_X^2(T - t).$$

If $\sigma_X(t)$ is time varying, then $V(t,T) = \int_t^T \sigma_X^2(s)ds$.

The Displaced Black Formula for pricing caplets and floorlets

If the simple forward rate has dynamics given by

$$dF(t, T_i, T_{i+1}) = \sigma_i(F(t, T_i, T_{i+1}) + \delta)dW^{pricing}(t),$$

the values at date t of caplets and floorlets with strike L_x and maturity date T_i on the simple forward rate $F(T_i, T_{i+1}, T_{i+1})$, are respectively

$$caplet(t) =$$
$$P(t, T_{i+1}) \times \left((F(t, T_i, T_{i+1}) + \delta) \times \mathcal{N}\left(d_1^i\right) - (L_x + \delta) \times \mathcal{N}\left(d_2^i\right) \right) \alpha_{i,i+1},$$
$$floorlet(t) =$$
$$P(t, T_{i+1}) \times \left((L_x + \delta) \times \mathcal{N}\left(-d_2^i\right) - (F(t, T_i, T_{i+1}) + \delta) \times \mathcal{N}\left(-d_1^i\right) \right) \alpha_{i,i+1},$$

where:

$$d_1^i = \frac{\ln\left(\frac{F(t, T_i, T_{i+1}) + \delta}{L_x + \delta}\right) + \frac{1}{2}V(t, T_i)}{\sigma\sqrt{T_i - t}}, d_2^i = d_1^i - \sigma\sqrt{T_i - t},$$

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz, V(t, T_i) = \sigma^2 \times (T_i - t).$$

Example (Pricing a 9x12 caplet using the Displaced Black formula)

- The 9m discount factor is 1.0010 (9m LIBOR is -0.131%). The 12m discount factor is 1.0015 (12m LIBOR is -0.203%).
- The caplet strike is -0.2%. The (constant) percentage volatility is 120%.
- The displacement coefficient is 1%.
- The 9x12 simple forward rate is

$$F = \frac{1}{0.25}\left(\frac{1.0010}{1.0015} - 1\right) = -0.2162\%.$$

- In addition

$$d_1 = \frac{\ln\left(\frac{-0.2162\%+1\%}{-0.2\%+1\%}\right) + \frac{1}{2} \times 1.2^2 \times 0.75}{\sqrt{1.2^2 \times 0.75}} = 0.49991,$$

and
$d_2 = d_1 - \sqrt{1.2^2 \times 0.75} = -0.53932, \mathcal{N}(d_1) = 0.691432, \mathcal{N}(d_2) = 0.294834.$

- Therefore the caplet price is 0.0007660 obtained as

$$1.0015 \times ((-0.2162\% + 1\%) \times 0.691432 - (-0.2\% + 1\%) \times 0.294834) \times 0.25.$$

Conclusions I

- Different solution methods were discussed to cope with negative interest rates.
- The forward rate has a normal distribution under Bachelier's model and it therewith allows negative forward rates.

+ It does not introduce an additional (shift) parameter.
+ Analytic expressions are available for the call and put price valuations and for its risk metrics.

- The only disadvantage of the normal models is that they assume a positive probability on large negative rates.

- Displaced models limit the largest negative rate.

+ It could be argued that there is a limit to how low rates can go, since storing (and transporting) cash becomes cheaper than the interest at a certain point.

Conclusions II

- For financial institutions this is different however, because they are obliged by regulation to have a certain part of their assets stored at the central bank.
  - The displacement parameter has to be chosen a priori. This choice is more of an art than a science.
  - When the rate decreases sufficiently, a larger displacement parameter needs to be introduced and all implied volatilities, valuations and risk metrics have to be adjusted.
- More sophisticated solutions (not presented here) are possible, such as the free boundary models.
  + These models can model rates from the entire real line and do not introduce an additional parameter.
  - Nevertheless, the models have an uncontrollable spike around zero in their probability distribution, while this is the most crucial area in a low or negative rate environment.
  - There is no accurate analytic (calibration) formula available for the free boundary SABR model.
  - The free boundary models do not have an obvious model interpretation and are therewith not intuitive for traders.

Black formula for Asset-or-Nothing Options

An Asset-or-Nothing option has payoff at time T_{i+1} equal to:

$$N \times \alpha_{T_i, T_{i+1}} \times L\left(T_i, T_{i+1}\right) \times \ \mathbf{1}_{L(T_i, T_{i+1}) > L_x}.$$

The Black price is given by

$$N \times \alpha_{T_i, T_{i+1}} \times P\left(t, T_{i+1}\right) \times F\left(t, T_i, T_{i+1}\right) \times \mathcal{N}\left(d_1^i\right).$$

where

$$d_1^i = \frac{\ln\left(\frac{F(t, T_i, T_{i+1})}{L_x}\right) + \frac{1}{2} V(t, T_i)}{\sqrt{V(t, T_i)}}, V(t, T_i) = \sigma^2 \times (T_i - t).$$

Black formula for Cash-or-Nothing

A Cash-or-Nothing option has payoff at time T_{i+1} equal to:

$$N \times \alpha_{T_i, T_{i+1}} \times \mathbf{1}_{L(T_i, T_{i+1}) > L_x}.$$

The Black price is given by

$$N \times \alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times \mathcal{N}\left(d_2^i\right).$$

where

$$d_2^i = d_1^i - \sqrt{V(t, T_i)}.$$

Displaced Black formula for Asset-or-Nothing Options

An Asset-or-Nothing option has payoff at time T_{i+1} equal to:

$$N \times \alpha_{T_i, T_{i+1}} \times L(T_i, T_{i+1}) \times \mathbf{1}_{L(T_i, T_{i+1}) > L_x}.$$

The Black price is given by

$$N \times \alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times (F(t, T_i, T_{i+1}) + \delta) \times \mathcal{N}\left(d_1^i\right).$$

where

$$d_1^i = \frac{\ln\left(\frac{F(t, T_i, T_{i+1}) + \delta}{L_x + \delta}\right) + \frac{1}{2} V(t, T_i)}{\sqrt{V(t, T_i)}}, V(t, T_i) = \sigma^2 \times (T_i - t).$$

Displaced Black formula for Cash-or-Nothing

A Cash-or-Nothing option has payoff at time T_{i+1} equal to:

$$N \times \alpha_{T_i, T_{i+1}} \times \mathbf{1}_{L(T_i, T_{i+1}) > L_x}.$$

The Black price is given by

$$N \times \alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times \mathcal{N}\left(d_2^i\right).$$

where

$$d_2^i = \frac{\ln\left(\frac{F(t, T_i, T_{i+1}) + \delta}{L_x + \delta}\right) - \frac{1}{2} V(t, T_i)}{\sqrt{V(t, T_i)}}, V(t, T_i) = \sigma^2 \times (T_i - t).$$

Bachelier formula for Asset-or-Nothing options

An Asset-or-Nothing option has payoff at time T_{i+1} equal to:

$$N \times \alpha_{T_i, T_{i+1}} \times L\left(T_i, T_{i+1}\right) \times \mathbf{1}_{L(T_i, T_{i+1}) > L_x}.$$

The Bachelier price is given by

$$N \times \alpha_{T_i, T_{i+1}} \times P\left(t, T_{i+1}\right) \times \left(F\left(t, T_i, T_{i+1}\right) \times \mathcal{N}\left(d_1^i\right) + \sqrt{V(t, T_i)}\, n(d_1^i)\right),$$

where

$$d_1^i = \frac{F\left(t, T_i, T_{i+1}\right) - K}{\sqrt{V(t, T_i)}}, V(t, T_i) = \sigma^2 \times \left(T_i - t\right).$$

Bachelier formula for Cash-or-Nothing

A Cash-or-Nothing option has payoff at time T_{i+1} equal to

$$N \times \alpha_{T_i, T_{i+1}} \times \mathbf{1}_{L(T_i, T_{i+1}) > L_x}.$$

The Bachelier price is given by

$$N \times \alpha_{T_i, T_{i+1}} \times P(t, T_{i+1}) \times \mathcal{N}\left(d_1^i\right).$$

Pricing Caps in Bloomberg

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March 25, 2020

References

Useful Readings

- **Bloomberg Terminal**: Book a place in the library.

Excel Files

- CAPPricingBloomberg.xlsm

Outline

1. **Command SWPM in Bloomberg**
2. **Select Contract Type**
3. **Setting CAP characteristics: Bachelier Model**
   - Pricing
   - Sensitivities
4. **Setting CAP characteristics: Black Model**
   - Pricing
   - Sensitivities
5. **Setting Swaptions characteristics: Bachelier Model**

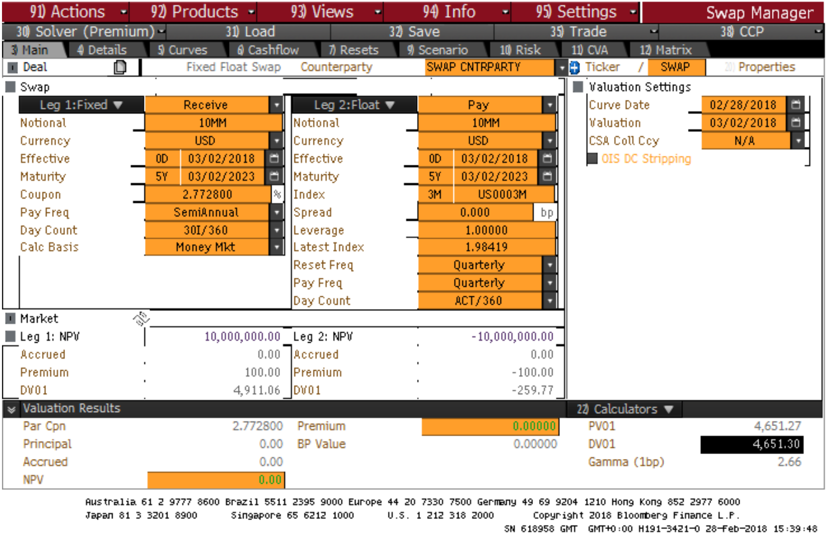
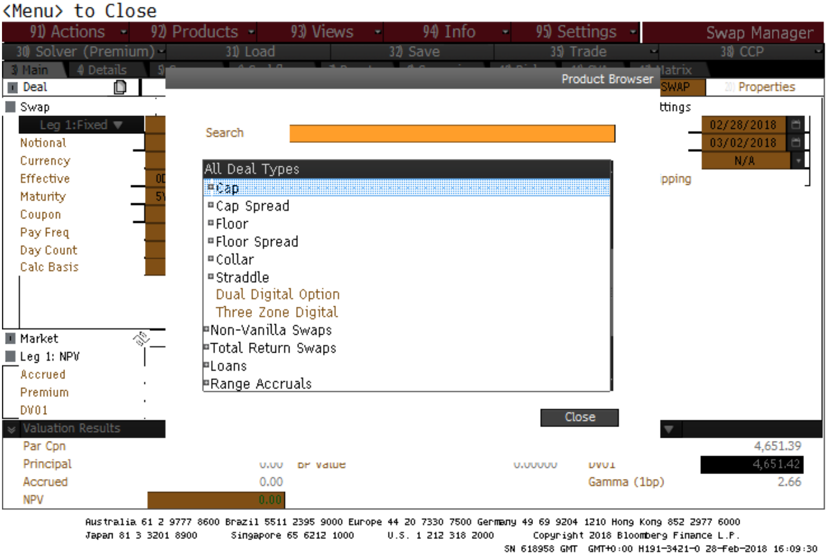


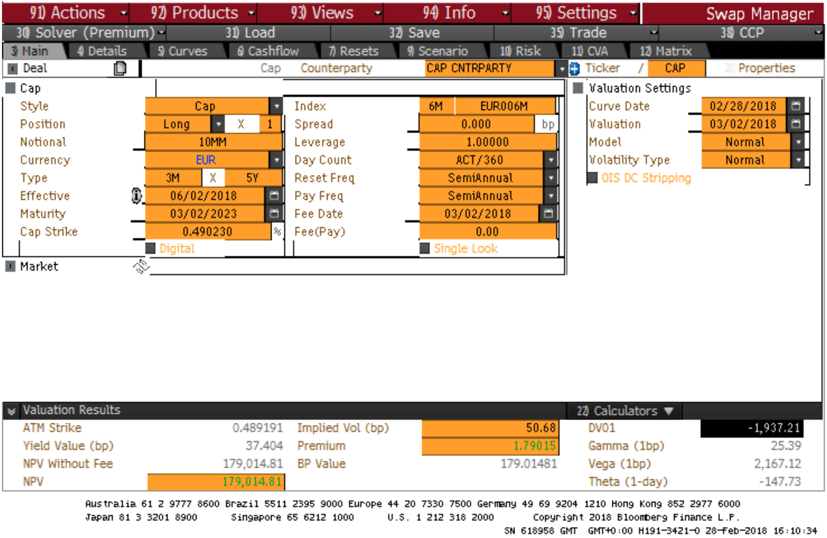
Figure 1: Swap Command in Bloomberg: select swap characteristics of an US Swap

CAP settings

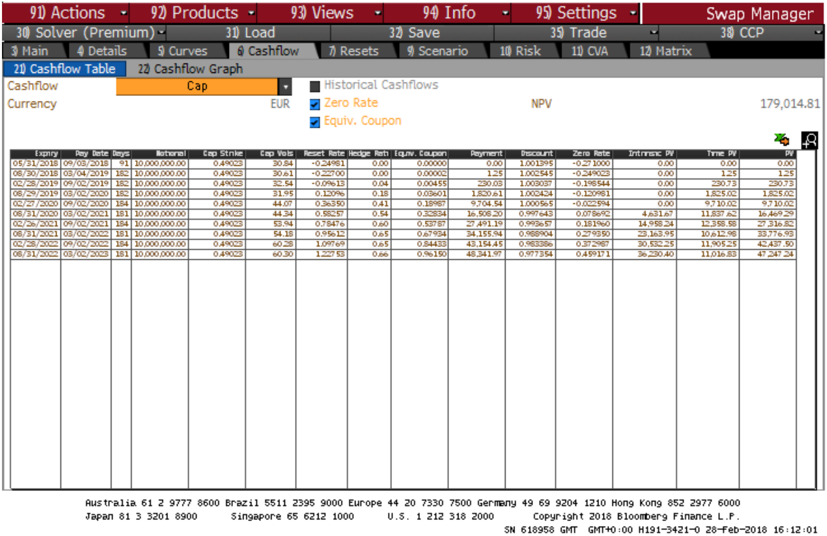


Pricing Caps with the Bachelier Model

Setting CAP characteristics: Bachelier Model



CAP Cash Flows with Bachelier Model



Expiry	Pay Date	Days	Notional	Cap Strike	Cap Vols	Reset Rate	Hedge Rati	Equiv. Coupon	Payment	Discount	Zero Rate	Intrinsic PV	Time PV	PV
05/31/2018	09/03/2018	91	10,000,000.00	0.49023	30.84	-0.24981	0.00	0.00000	0.00	1.001395	-0.271000	0.00	0.00	0.00
08/30/2018	03/04/2019	182	10,000,000.00	0.49023	30.61	-0.22700	0.00	0.00002	1.25	1.002545	-0.249023	0.00	1.25	1.25
02/28/2019	09/02/2019	182	10,000,000.00	0.49023	32.54	-0.09613	0.04	0.00455	230.03	1.003037	-0.198544	0.00	230.73	230.73
08/29/2019	03/02/2020	182	10,000,000.00	0.49023	31.95	0.12096	0.18	0.03601	1,820.61	1.002424	-0.120981	0.00	1,825.02	1,825.02
02/27/2020	09/02/2020	184	10,000,000.00	0.49023	44.07	0.36350	0.41	0.18987	9,704.54	1.000565	-0.022594	0.00	9,710.02	9,710.02
08/31/2020	03/02/2021	181	10,000,000.00	0.49023	44.34	0.58257	0.54	0.32834	16,508.20	0.997643	0.078692	4,631.67	11,837.62	16,469.29
02/26/2021	09/02/2021	184	10,000,000.00	0.49023	53.94	0.78476	0.60	0.53787	27,491.19	0.993657	0.181960	14,958.24	12,358.58	27,316.82
08/31/2021	03/02/2022	181	10,000,000.00	0.49023	54.18	0.95612	0.65	0.67934	34,155.94	0.988904	0.279350	23,163.95	10,612.98	33,776.93
02/28/2022	09/02/2022	184	10,000,000.00	0.49023	60.28	1.09769	0.65	0.84433	43,154.45	0.983386	0.372987	30,532.25	11,905.25	42,437.50
08/31/2022	03/02/2023	181	10,000,000.00	0.49023	60.30	1.22753	0.66	0.96150	48,341.97	0.977354	0.459171	36,230.40	11,016.83	47,247.24

Figure 4: Cash flows of a cap

Example (Reconstruct the caplets in Bloomberg)

- The Cap is composed of different caplets. Let us consider the last caplet.
- Valuation Date: 2nd March 2018;
- Caplet Maturity Date: 31st Aug, 2022;
- Caplet Payment Date: 2nd March, 2023;
- Caplet Tenor α_{T_1,T_2}: 0.50278=(181 days/360)
- Caplet Time to Maturity $T_1 - t$: 4.50137, that is computed using the ACT/ACT ISDA convention (see accompanying Excel file for exact calculations)
- Forward Rate **F**: 1.22753%, computed from the discount factors relative to the Maturity and Payment Dates.
- Caplet Strike **K**: 0.49023%;
- Forward Normal Volatility σ: 0.6030% =(quoted vol/10000);
- Notional **N**: 10,000,000;
- Discount Factor to the Maturity date $P(t, T_1)$: 0.983386.
- Discount Factor to the Payment date $P(t, T_2)$: 0.977354.

Example (Reconstruct the caplets in Bloomberg)

- We can apply the Bachelier formula

$$P(t, T_2) \times \left((F - K)\,\mathcal{N}(d_1) + \sqrt{V(t, T_1)}\, n(d_1) \right) \times \alpha_{T_2 - T_1} \times N,$$

where

$$V(t, T_1) = \sigma^2 \times (T_1 - t) = (0.6030\%)^2 \times 4.50137 = 0.000163674,$$

and

$$d_1 = \frac{F - K}{\sqrt{V(t, T_1)}} = \frac{1.22753\% - 0.49023\%}{\sqrt{0.000163674}} = \frac{0.73730\%}{0.012794} = 0.57631.$$

- Therefore

$$\mathcal{N}(0.57631) = 0.71780, \text{ and } n(0.57631) = 0.33790.$$

- Finally, the caplet price (column PV) is **47,248.54**

$$= 0.977354 \times (0.7373\% \times 0.7178 + 0.012794 \times 0.3379) \times 0.50278 \times N.$$

Example (Intrinsic and Time Value)

- In addition, we have the Intrinsic Present Value of the caplet is

$$P(t, T_2) \times \max(F - K; 0) \times \alpha_{T_1,T_2} \times N,$$

and we obtain

$$0.977354 \times \max(1.22753\% - 0.49023\%; 0) \times 0.50278 \times N = \mathbf{36230.3}.$$

- By difference with the caplet price we also obtain the Time Value

$$47,248.54 - 36230.3 = \mathbf{11018.2}.$$

- The Fair Value of the cap (Cap NPV) is obtained by summing the PV of each caplet

$$0.00 + 1.25 + 230.85 + \cdots + 33,778.11 + 42,438.37 + 47,248.54 = \mathbf{179,019.36}$$

Example (Cap Implied Volatility)

- The Cap Implied Volatility is then obtained by pricing all the caplets using the same volatility (so called **Flat Volatility**) mantaining constant the value of the cap.
- This amounts to solve a non-linear equation, returning as solution

Table 1: From forward volatilities to flat volatility.

Payment Date	Cap Vol (bps)	Caplet Price	Flat Vol	Caplet Price
03 September 2018	30.84	0.00	50.68	3.00
04 March 2019	30.61	1.25	50.68	149.62
02 September 2019	32.54	230.85	50.68	1,559.20
02 March 2020	31.95	1,825.55	50.68	5,325.41
02 September 2020	44.07	9,709.90	50.68	11,578.54
02 March 2021	44.34	16,470.18	50.68	18,461.15
02 September 2021	53.94	27,316.62	50.68	26,232.76
02 March 2022	54.18	33,778.11	50.68	32,617.31
02 September 2022	60.28	42,438.37	50.68	39,128.68
02 March 2023	60.30	47,248.54	50.68	43,963.71
	CAP NPV	179,019.36		179,019.36

Example (Sensitivities: Cap DV01)

- We compute the sensitivity of the CAP to a shift in the forward curve. In practice, the DV01 of each caplet is

$$-P(t, T_i) \times N(d_1) \times \alpha_{T_{i-1}, T_i} \times N \times 0.01\%$$

- With reference to the caplet expiring in March 2023, we obtain

$$0.977354 \times 0.71780 \times 0.502777778 \times 10,000,000 = -352.719.$$

- If we sum the DV01 of the different Caplets we obtain the CAP DV01=

$$0.000 - 0.222 - 17.945 + \cdots - 336.642 - 348.256 - 352.719 = -\mathbf{1951}.338$$

Example (Sensitivities: Cap Gamma)

- We compute the second order sensitivity of the CAP to a shift in the forward curve.
- In practice, the Gamma of each caplet is

$$-P(t, T_i) \times \frac{n(d_1)}{\sqrt{V(t, T_{i-1})}} \times \alpha_{T_{i-1}, T_i} \times N \times 0.01\%$$

- With reference to the caplet expiring in March 2023, we obtain

$$0.977354 \times \frac{0.33790}{\sqrt{0.01279}} \times 0.502777778 \times 10,000,000 \times (0.01\%)^2 = 1.298.$$

- If we sum the Gamma of the different caplets we obtain the CAP Gamma

$$0.000 + 0.037 + 1.218 + \cdots + 1.760 + 1.465 + 1.298 = \mathbf{17.198}$$

(Bloomberg gives a sligthly different value of 25.39)

Example (Sensitivities: Cap Vega)

- Let us reprice the cap by shifting up by 1 basis points all cap vols (so for example with reference to our caplet we use a volatility of 60.3001 basis points instead of 60.30).
- We obtain a CAP NPV equal to 179,019.58.
- The cap vega is estimated as forward difference

$$\frac{179,019.58 - 179,019.36}{0.0001} = 2166.41.$$

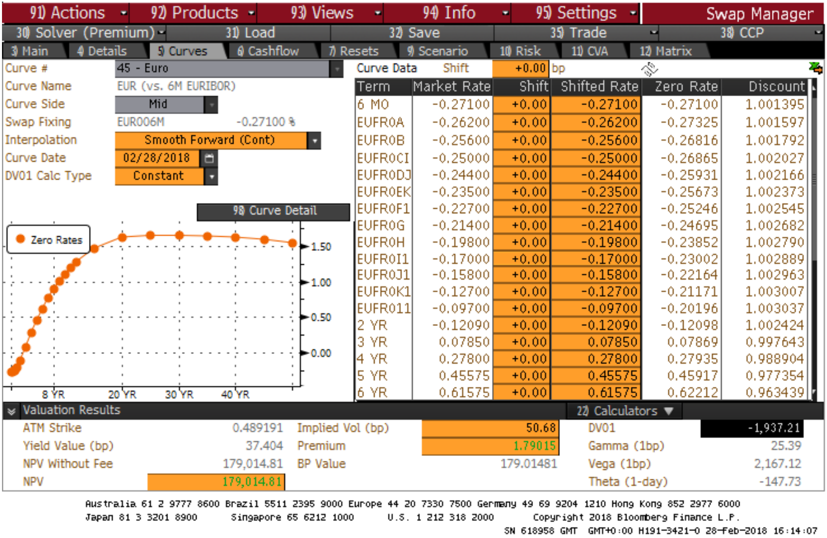
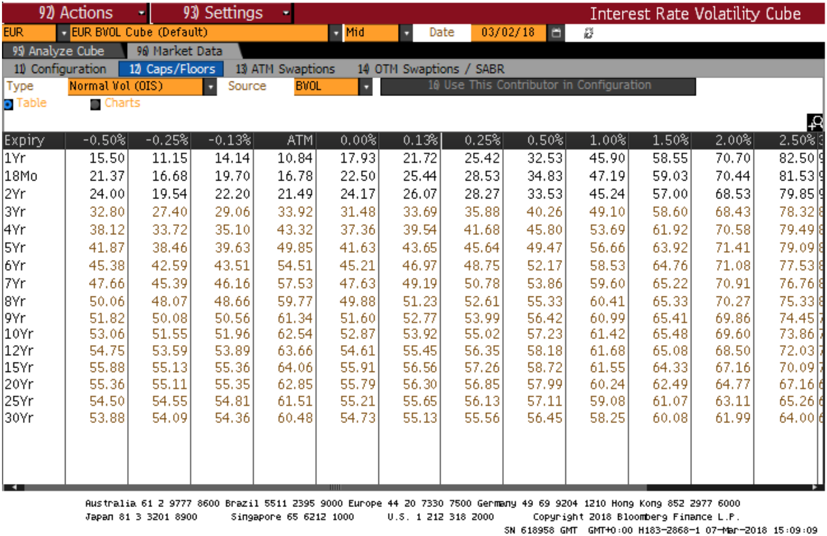


Figure 5: Settings for the construction of the discount curve in pricing the cap

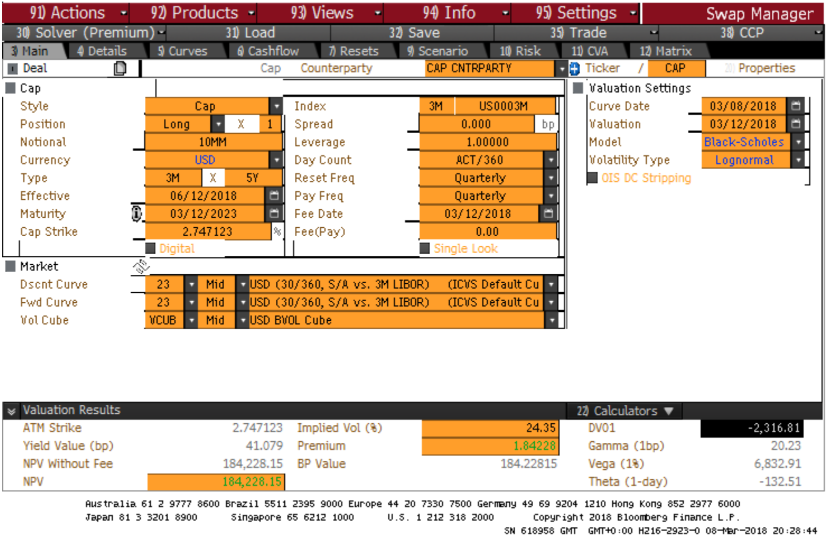


Expiry	-0.50%	-0.25%	-0.13%	ATM	0.00%	0.13%	0.25%	0.50%	1.00%	1.50%	2.00%	2.50%
1Yr	15.50	11.15	14.14	10.84	17.93	21.72	25.42	32.53	45.90	58.55	70.70	82.50
18Mo	21.37	16.68	19.70	16.78	22.50	25.44	28.53	34.83	47.19	59.03	70.44	81.53
2Yr	24.00	19.54	22.20	21.49	24.17	26.07	28.27	33.53	45.24	57.00	68.53	79.85
3Yr	32.80	27.40	29.06	33.92	31.48	33.69	35.88	40.26	49.10	58.60	68.43	78.32
4Yr	38.12	33.72	35.10	43.32	37.36	39.54	41.68	45.80	53.69	61.92	70.58	79.49
5Yr	41.87	38.46	39.63	49.85	41.63	43.65	45.64	49.47	56.66	63.92	71.41	79.09
6Yr	45.38	42.59	43.51	54.51	45.21	46.97	48.75	52.17	58.53	64.76	71.08	77.53
7Yr	47.66	45.39	46.16	57.53	47.63	49.19	50.78	53.86	59.60	65.22	70.91	76.76
8Yr	50.06	48.07	48.66	59.77	49.88	51.23	52.61	55.33	60.41	65.33	70.27	75.33
9Yr	51.82	50.08	50.56	61.34	51.60	52.77	53.99	56.42	60.99	65.41	69.86	74.45
10Yr	53.06	51.55	51.96	62.54	52.87	53.92	55.02	57.23	61.42	65.48	69.60	73.86
12Yr	54.75	53.59	53.89	63.66	54.61	55.45	56.35	58.18	61.68	65.08	68.50	72.03
15Yr	55.88	55.13	55.36	64.06	55.91	56.56	57.26	58.72	61.55	64.33	67.16	70.09
20Yr	55.36	55.11	55.35	62.85	55.79	56.30	56.85	57.99	60.24	62.49	64.77	67.16
25Yr	54.50	54.55	54.81	61.51	55.21	55.65	56.13	57.11	59.08	61.07	63.11	65.26
30Yr	53.88	54.09	54.36	60.48	54.73	55.13	55.56	56.45	58.25	60.08	61.99	64.00

Figure 6: Settings for the construction of the volatility surface in pricing the cap

Pricing Caps with the Black Model

Setting CAP characteristics: Black Model



CAP Cash Flows with Black Model

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91) Actions | 92) Products | 93) Views | 94) Info | 95) Settings | Swap Manager

30) Solver (Premium) | 31) Load | 32) Save | 35) Trade | 38) CCP

3) Main | 4) Details | 5) Curves | 6) Cashflow | 7) Resets | 9) Scenario | 10) Risk | 11) CVA | 12) Matrix

21) Cashflow Table | 22) Cashflow Graph

Cashflow: Cap — Historical Cashflows

Currency: USD — Zero Rate — NPV 184,228.15

Equiv. Coupon

Expiry	Pay Date	Days	Notional	Cap Strike	Cap Vols	Reset Rate	Hedge Ratio	Payment	Discount	Intrinsic PV	Time PV	PV
06/08/2018	09/12/2018	92	10,000,000.00	2.74712	14.14	2.28957	0.01	6.04	0.988983	0.00	5.97	5.97
09/10/2018	12/12/2018	91	10,000,000.00	2.74712	13.71	2.34238	0.06	129.22	0.983162	0.00	127.04	127.04
12/10/2018	03/12/2019	90	10,000,000.00	2.74712	13.07	2.49502	0.22	810.04	0.977068	0.00	791.46	791.46
03/08/2019	06/12/2019	92	10,000,000.00	2.74712	17.97	2.59146	0.40	3,122.68	0.970639	0.00	3,031.00	3,031.00
06/10/2019	09/12/2019	92	10,000,000.00	2.74712	18.35	2.69773	0.50	5,067.04	0.963993	0.00	4,884.60	4,884.60
09/10/2019	12/12/2019	91	10,000,000.00	2.74712	18.70	2.74718	0.54	6,327.35	0.957345	1.39	6,056.07	6,057.46
12/10/2019	03/12/2020	91	10,000,000.00	2.74712	19.09	2.78013	0.56	7,441.41	0.950665	793.23	6,281.06	7,074.29
03/10/2020	06/12/2020	92	10,000,000.00	2.74712	23.42	2.81442	0.59	10,224.26	0.943876	1,623.18	8,027.26	9,650.43
06/10/2020	09/14/2020	94	10,000,000.00	2.74712	23.48	2.84180	0.59	11,480.98	0.936924	2,316.22	8,440.58	10,756.80
09/10/2020	12/14/2020	91	10,000,000.00	2.74712	23.49	2.85884	0.60	11,913.78	0.930201	2,626.86	8,455.35	11,082.21
12/10/2020	03/12/2021	88	10,000,000.00	2.74712	23.54	2.86648	0.60	12,140.35	0.923730	2,695.17	8,519.23	11,214.40
03/10/2021	06/14/2021	94	10,000,000.00	2.74712	25.78	2.86704	0.60	14,566.30	0.916866	2,870.76	10,484.59	13,355.35
06/10/2021	09/13/2021	91	10,000,000.00	2.74712	25.80	2.86641	0.61	14,607.64	0.910271	2,744.75	10,552.16	13,296.91
09/09/2021	12/13/2021	91	10,000,000.00	2.74712	25.82	2.86689	0.61	15,104.65	0.903722	2,736.07	10,914.33	13,650.40
12/09/2021	03/14/2022	91	10,000,000.00	2.74712	25.83	2.86848	0.61	15,599.34	0.897216	2,752.24	11,243.74	13,995.98
03/10/2022	06/13/2022	91	10,000,000.00	2.74712	28.70	2.87124	0.61	17,648.56	0.890751	2,794.64	12,925.83	15,720.48
06/09/2022	09/12/2022	91	10,000,000.00	2.74712	28.81	2.87520	0.61	18,252.70	0.884325	2,862.94	13,278.38	16,141.31
09/08/2022	12/12/2022	91	10,000,000.00	2.74712	28.82	2.88039	0.61	18,806.74	0.877932	2,957.51	13,553.53	16,511.04
12/08/2022	03/13/2023	91	10,000,000.00	2.74712	28.82	2.88711	0.61	19,368.47	0.871572	3,084.16	13,796.85	16,881.01

Example (Reconstruct the caplets in Bloomberg)

- The US Cap is composed of different caplets having quarterly cash flows. Let us consider the last caplet.
- Valuation Date: 12th of March 2018;
- Caplet (Reset) Maturity Date: 8th December, 2022;
- Caplet Start Date ($\boldsymbol{T_1}$): 12th December, 2023;
- Caplet Payment Date ($\boldsymbol{T_2}$): 13th March, 2023;
- Caplet Tenor (α_{T_1,T_2}): 0.252778=(91 days/360)
- Caplet Time to Maturity ($\boldsymbol{T_1 - t}$): 4.74521, that is computed using the ACT/ACT ISDA convention (see accompanying Excel file for exact calculations).
- Forward Rate $\boldsymbol{(F(t, T_1, T_2))}$: 2.88711%, computed from the discount factors relative to the Maturity and Payment Dates.
- Caplet Strike $\boldsymbol{K}$: 2.74712%;
- Forward Lognormal Volatility (σ): 28.82%;
- Notional ($\boldsymbol{N}$): 10,000,000;
- Discount Factor to the Maturity date $\boldsymbol{P(t, T_1)}$: 0.871572;

Example (Reconstruct the caplets in Bloomberg)

- We can apply the Black formula

$$P(t, T_2) \times (F(t, T_1, T_2)\mathcal{N}(d_1) - K\mathcal{N}(d_2)) \times \alpha_{T_1,T_2} \times N,$$

where

$$V(t, T_1) = \sigma^2 \times (T_1 - t) = (0.2882)^2 \times 4.74521 = 0.394133,$$

and

$$d_1 = \frac{\ln\left(\frac{2.88711\%}{2.74712\%}\right) + \frac{1}{2}0.394133}{\sqrt{0.394133}} = 0.39307, \text{ and } d_2 = -0.23473.$$

- Therefore

$$\mathcal{N}(0.39307) = 0.65287, \text{ and } \mathcal{N}(-0.234730) = 0.40721.$$

- Finally, the caplet price (column PV) is **16,881.41** (Bloomberg gives 16881.01)

$$= 0.871572 \times (2.88711\% \times 0.65287 - 2.74712\% \times 0.40721) \times 0.25278 \times N.$$

Example (Intrinsic and Time Value)

- In addition, the Intrinsic Present Value of the caplet is

$$P(t, T_2) \times \max\left(F(t, T_1, T_2) - K; 0\right) \times \alpha_{T_2 - T_1} \times N,$$

and we obtain

$$0.871572 \times \max\left(2.88711\% - 2.74712\%; 0\right) \times 0.25278 \times N = \mathbf{3084.2}.$$

- By difference with the caplet price we also obtain the Time Value

$$16881.43 - 3084.2 = \mathbf{13797.26}.$$

- The Fair Value of the cap (Cap NPV) is obtained by summing the PV of each caplet

$$5.96 + 126.95 + 791.94 + \cdots + 16141.37 + 16511.74 + 16881.43 = \mathbf{184,219.6}.$$

Example (Cap Implied Volatility)

- The Cap Implied Volatility is then obtained by pricing all the caplets using the same volatility (so called **Flat Volatility**) mantaining constant the value of the cap.
- This amounts to solve a non-linear equation, returning as solution 24.25%.

Table 2: Pricing a cap using forward forward vols and flat volatility

Payment Date	Cap Vol (bps)	Caplet Price	Flat Vol	Caplet Price
12 September 2018	14.14	5.96	24.35	210.16
12 December 2018	13.71	126.95	24.35	1033.86
12 March 2019	13.07	791.94	24.35	2839.19
12 June 2019	17.97	3031.98	24.35	4634.48
12 September 2019	18.35	4885.19	24.35	6660.33
12 December 2019	18.7	6058.89	24.35	7878.26
12 March 2020	19.09	7073.26	24.35	8896.10
12 June 2020	23.42	9650.50	24.35	9997.31
14 September 2020	23.48	10754.15	24.35	11103.75
14 December 2020	23.49	11077.44	24.35	11427.72
12 March 2021	23.54	11209.42	24.35	11541.47
14 June 2021	25.78	13356.10	24.35	12712.36
13 September 2021	25.8	13296.30	24.35	12644.17
13 December 2021	25.82	13650.50	24.35	12970.49
14 March 2022	25.83	13994.83	24.35	13292.38
13 June 2022	28.7	15721.64	24.35	13612.28
12 September 2022	28.81	16141.37	24.35	13931.96
12 December 2022	28.82	16511.74	24.35	14252.91
13 March 2023	28.82	16881.43	24.35	14580.38
	CAP NPV	**184,219.58**		**184,219.57**

Example (Sensitivities: Cap DV01)

- We compute the sensitivity of the CAP to a shift in the forward curve. In practice, the DV01 of each caplet is

$$-P(t, T_2) \times \mathcal{N}(d_1) \times \alpha_{T_1,T_2} \times N \times 0.01\%$$

- With reference to the caplet expiring in Dec. 2022, we obtain

$$0.871572 \times 0.64018 \times 0.252778 \times 10,000,000 \times 0.01\% = -141.04.$$

- If we sum the DV01 of the different Caplets we obtain the CAP DV01=

$$-18.10 - 49.76 - \cdots - 141.32 - 141.04 = -\mathbf{2,394.84}.$$

(Bloomberg gives a sligthly different value of 2316.81).

Example (Sensitivities: Cap Gamma)

- We compute the second order sensitivity of the CAP to a shift in the forward curve.
- In practice, the Gamma of each caplet is

$$\Gamma = \frac{P(t, T_2)}{\sigma\sqrt{T_1 - t} \times F(t, T_1, T_2)} \times n(d_1) \times \alpha_{T_1, T_2} \times N \times (0.01)^2,$$

where $n(.)$ is the standard normal density function.

- With reference to the caplet expiring in December 2022, we obtain

$$\frac{0.871572}{\sqrt{0.2814187}} \times n(0.3589372) \times 0.252778 \times 10,000,000 \times (0.01\%)^2 = 0.5381.$$

- If we sum the Gamma of the different caplets we obtain the CAP Gamma

$$1.2615 + 1.7284 + 1.7432 + \cdots + 0.5830 + 0.5599 + 0.5381 = \mathbf{18.31}$$

(Bloomberg gives a sligthly different value of 20.23).

Example (Sensitivities: Cap Vega)

- The Vega, i.e. the first derivative of the caplet price with respect to the volatility parameter is given by:

$$\upsilon = P(t, T_2) \times \sqrt{T_1 - t} \times F(t, T_1, T_2) \times n(d_1) \times \alpha_{T_1, T_2} \times N \times 1\%.$$

- With reference to the caplet under examination, we have

$$0.871572 \times \sqrt{4.74521} \times 2.88711\% \times n(0.358937) \times 0.252778 \times N \times 1\% = 518.2$$

- Summing the vega across caplets with different maturities, we obtain a CAP vega equal to 7,008 (Bloomberg gives 6,832).
- An approximate vega estimate is obtained by repricing the cap by shifting up by 1 basis points all cap vols (so for example with reference to our caplet we use a flat volatility of 24.36% instead of 24.35%).
- We obtain a CAP NPV equal to 184,289.65. The cap vega is estimated as forward difference

$$\frac{184,289.65 - 184,219.58}{0.0001} = 7007,$$

not very different from the 7008 estimate obtained above.

Volatility Surface for the Black Model

92) Actions | 93) Settings | Interest Rate Volatility Cube

USD | USD Bloomberg Cube | Mid | Date 03/07/18

95) Analyze Cube | 96) Market Data

11) Configuration | 12) Caps/Floors | 13) ATM Swaptions | 14) OTM Swaptions / SABR

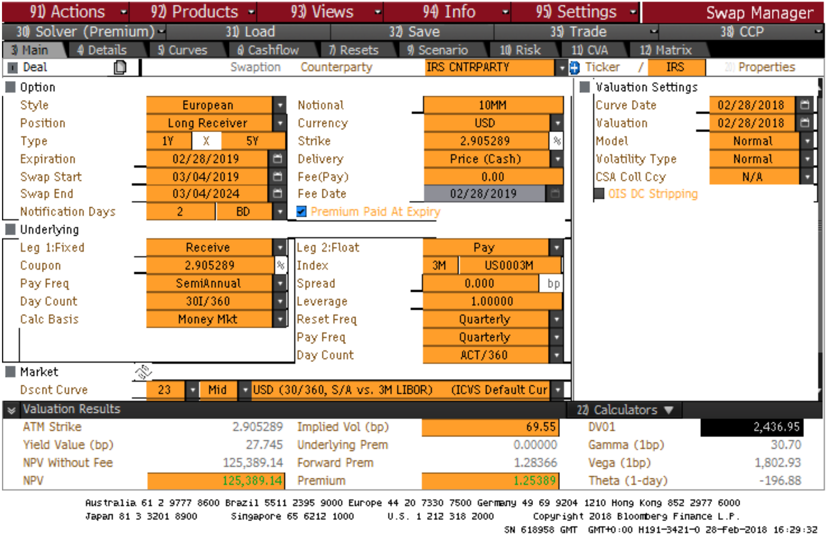
Type: Black Vol (IBOR) | Source: BBIR | 16) Use This Contributor in Configuration

Table | Charts

Expiry	ATM	1.00%	1.50%	2.00%	3.00%	3.50%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	10.00%	11.00%	12.00%	13.
1Yr	13.94	34.81	23.98	16.65	14.69	17.68	20.37	24.68	27.98	30.64	32.84	34.73	36.36	37.80	39.08	40.
2Yr	15.82	36.33	27.00	19.87	16.59	18.20	19.88	22.87	25.34	27.39	29.12	30.61	31.92	33.07	34.11	35.
3Yr	19.35	41.86	31.64	24.07	19.60	20.56	21.75	24.12	26.16	27.88	29.33	30.58	31.67	32.63	33.49	34.
4Yr	21.75	43.06	33.48	26.51	21.74	22.36	23.30	25.26	26.99	28.47	29.73	30.82	31.77	32.62	33.37	34.
5Yr	23.60	44.20	34.96	28.36	23.48	23.84	24.55	26.15	27.64	28.93	30.05	31.02	31.88	32.64	33.32	33.
6Yr	24.54	41.95	34.26	28.74	24.40	24.63	25.15	26.36	27.47	28.43	29.27	30.00	30.66	31.25	31.78	32.
7Yr	25.32	43.06	35.12	29.60	25.15	25.25	25.72	26.94	28.16	29.29	30.31	31.23	32.08	32.85	33.57	34.
8Yr	25.05	42.26	34.54	29.27	24.85	24.80	25.13	26.16	27.26	28.30	29.25	30.11	30.89	31.61	32.27	32.
9Yr	25.10	42.31	34.57	29.37	24.88	24.70	24.91	25.78	26.76	27.71	28.59	29.39	30.12	30.79	31.40	31.
10Yr	25.81	42.77	35.13	30.03	25.62	25.43	25.66	26.61	27.72	28.80	29.81	30.73	31.60	32.38	33.11	33.
12Yr	25.61	39.75	33.44	29.24	25.49	25.27	25.42	26.11	26.93	27.75	28.52	29.23	29.90	30.52	31.09	31.
15Yr	25.48	42.67	34.79	29.86	25.35	24.86	24.84	25.39	26.20	27.03	27.81	28.54	29.22	29.84	30.41	30.
20Yr	24.72	41.77	33.92	29.10	24.63	24.07	24.03	24.60	25.47	26.37	27.23	28.03	28.77	29.44	30.07	30.
25Yr	24.78	42.00	33.97	29.14	24.66	24.06	23.97	24.49	25.31	26.18	27.01	27.78	28.48	29.14	29.74	30.
30Yr	24.85	42.50	34.14	29.22	24.69	24.06	23.95	24.43	25.23	26.07	26.88	27.63	28.32	28.96	29.55	30.

Pricing Swaption with the Black Model

Swaptions in the Bachelier Model



Swaptions in the Bachelier Model

91) Actions | 92) Products | 93) Views | 94) Info | 95) Settings | Swap Manager

30) Solver (Premium) | 31) Load | 32) Save | 35) Trade | 38) CCP

3) Main | 4) Details | 5) Curves | 6) Cashflow | 7) Resets | 9) Scenario | 10) Risk | 11) CVA | 12) Matrix

21) Cashflow Table | 22) Cashflow Graph

Cashflow: Leg 1: Receive Fixed — Historical Cashflows — Accrued 0.00

Currency: USD — Zero Rate — NPV 0.00

Equiv. Coupon

Pay Date	Accrual Start	Accrual End	Days	Notional	Principal	Payment	Discount	PV
03/04/2019					-10,000,000.00	-10,000,000.00	0.976811	-9,768,114.20
09/04/2019	03/04/2019	09/04/2019	180	10,000,000.00	0.00	145,264.43	0.963675	139,987.65
03/04/2020	09/04/2019	03/04/2020	180	10,000,000.00	0.00	145,264.43	0.950324	138,048.24
09/04/2020	03/04/2020	09/04/2020	180	10,000,000.00	0.00	145,264.43	0.936675	136,065.58
03/04/2021	09/04/2020	03/04/2021	180	10,000,000.00	0.00	145,264.43	0.923247	134,114.87
09/07/2021	03/04/2021	09/07/2021	183	10,000,000.00	0.00	147,685.50	0.909548	134,327.04
03/04/2022	09/07/2021	03/04/2022	177	10,000,000.00	0.00	142,843.35	0.896707	128,088.64
09/06/2022	03/04/2022	09/06/2022	182	10,000,000.00	0.00	146,878.48	0.883487	129,765.25
03/06/2023	09/06/2022	03/06/2023	180	10,000,000.00	0.00	145,264.43	0.870771	126,492.04
09/05/2023	03/06/2023	09/05/2023	179	10,000,000.00	0.00	144,457.40	0.858020	123,947.38
03/04/2024	09/05/2023	03/04/2024	179	10,000,000.00	10,000,000.00	10,144,457.40	0.845514	8,577,277.49

Swaptions in the Bachelier Model

91) Actions | 92) Products | 93) Views | 94) Info | 95) Settings | Swap Manager

30) Solver (Premium) | 31) Load | 32) Save | 35) Trade | 38) CCP

3) Main | 4) Details | 5) Curves | 6) Cashflow | 7) Resets | 9) Scenario | 10) Risk | 11) CVA | 12) Matrix

21) Cashflow Table | 22) Cashflow Graph

Cashflow: Leg 2: Pay Float — Historical Cashflows — Accrued 0.00

Currency: USD — Zero Rate — NPV 0.00

Equiv. Coupon

Pay Date	Accrual Start	Accrual End	Days	Notional	Principal	Reset Date	Reset Rate	Payment	Discount	PV
03/04/2019					10,000,000.00			10,000,000.00	0.976811	9,768,114.20
06/04/2019	03/04/2019	06/04/2019	92	-10,000,000.00	0.00	02/28/2019	2.60967	-66,691.52	0.970340	-64,713.46
09/04/2019	06/04/2019	09/04/2019	92	-10,000,000.00	0.00	05/31/2019	2.70651	-69,166.41	0.963675	-66,653.92
12/04/2019	09/04/2019	12/04/2019	91	-10,000,000.00	0.00	09/02/2019	2.75339	-69,599.66	0.957014	-66,607.84
03/04/2020	12/04/2019	03/04/2020	91	-10,000,000.00	0.00	12/02/2019	2.78497	-70,397.94	0.950324	-66,900.84
06/04/2020	03/04/2020	06/04/2020	92	-10,000,000.00	0.00	03/02/2020	2.82453	-72,182.33	0.943513	-68,104.99
09/04/2020	06/04/2020	09/04/2020	92	-10,000,000.00	0.00	06/02/2020	2.85669	-73,004.29	0.936675	-68,381.31
12/04/2020	09/04/2020	12/04/2020	91	-10,000,000.00	0.00	09/02/2020	2.87758	-72,738.93	0.929911	-67,640.74
03/04/2021	12/04/2020	03/04/2021	90	-10,000,000.00	0.00	12/02/2020	2.88746	-72,186.61	0.923247	-66,646.04
06/04/2021	03/04/2021	06/04/2021	92	-10,000,000.00	0.00	03/02/2021	2.88882	-73,825.42	0.916481	-67,659.56
09/07/2021	06/04/2021	09/07/2021	95	-10,000,000.00	0.00	06/02/2021	2.88836	-76,220.66	0.909548	-69,326.34
12/06/2021	09/07/2021	12/06/2021	90	-10,000,000.00	0.00	09/03/2021	2.88668	-72,167.01	0.903031	-65,169.05
03/04/2022	12/06/2021	03/04/2022	88	-10,000,000.00	0.00	12/02/2021	2.88510	-70,524.59	0.896707	-63,239.90
06/06/2022	03/04/2022	06/06/2022	94	-10,000,000.00	0.00	03/02/2022	2.88473	-75,323.53	0.890003	-67,038.18
09/06/2022	06/06/2022	09/06/2022	92	-10,000,000.00	0.00	06/02/2022	2.88599	-73,753.17	0.883487	-65,159.98
12/05/2022	09/06/2022	12/05/2022	90	-10,000,000.00	0.00	09/02/2022	2.89024	-72,256.06	0.877149	-63,379.35
03/06/2023	12/05/2022	03/06/2023	91	-10,000,000.00	0.00	12/01/2022	2.89778	-73,249.51	0.870771	-63,783.54
06/05/2023	03/06/2023	06/05/2023	91	-10,000,000.00	0.00	03/02/2023	2.90757	-73,496.91	0.864418	-63,532.03
09/05/2023	06/05/2023	09/05/2023	92	-10,000,000.00	0.00	06/01/2023	2.91757	-74,560.07	0.858020	-63,974.06
12/04/2023	09/05/2023	12/04/2023	90	-10,000,000.00	0.00	09/01/2023	2.92670	-73,167.39	0.851788	-62,323.10
03/04/2024	12/04/2023	03/04/2024	91	-10,000,000.00	-10,000,000.00	11/30/2023	2.93567	-10,074,207.16	0.845514	-8,517,879.95

Example (Swaption characteristics)

The swaption has the following characteristics

- Valuation Date: Feb. 28, 2018;
- Swaption Expiration Date: Feb. 28 2019 (365 days);
- Underlying Swap Starts: 4th March, 2019;
- Underlying Swap Ends: 4th March, 2024;
- Fixed Leg Frequency: S/A, 30/360;
- Floating Leg Frequency: Q, ACT/360;
- Strike (ATM): 2.905289%;
- Forward Swap Normal Volatility σ: 0.6955% =(quoted vol/10000);
- Notional **N**: 10,000,000;

Example (Pricing the swaption)

- We can apply the Bachelier formula

$$\sum_{i=1}^{n} P(t, T_i)\alpha_{i-1,i} \times \left((S-K)\,\mathcal{N}(d_1) + \sqrt{V(t,T_1)}\,n(d_1) \right) \times \alpha_{T_2-T_1} \times N,$$

where

$$V(t, T_1) = \sigma^2 \times (T_1 - t) = (0.6955\%)^2 \times 1 = 0.0000484,$$

and

$$d_1 = \frac{S-K}{\sqrt{V(t,T_1)}} = \frac{2.905289\% - 2.905289\%}{\sqrt{0.0000484}} = 0.$$

- Therefore $\mathcal{N}(0) = 0.5$, and $n(0) = 0.398942$. The forward swap rate is as the ATM strike. The annuity value is equal to 4.51927 and has been computed referring to the discount factors and the accrual factors of the fixed leg.
- Finally, the Swaption price (column PV) is **125,393.55**

$$= 4.51927 \times \left((2.9053\% - 2.9053\%) \times 0.5 + \sqrt{0.0000484} \times 0.398942 \right) \times N.$$

Pricing of Structured Products

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SMM269 Fixed Income
Academic Year 2019-20

Main References:

Excel Files

- FI_CaseStudyPricingStructBond.xlsm

Outline

1. The Corporate Bond
2. Data Collection
3. Coupon Decomposition
4. The Pricing Procedure
   - Bootstrapping the discount curve
   - Interpolating the discount curve
   - Interpolating Volatility
   - Pricing
   - Sensitivitities & Hedging
   - Pricing a defaultable bond
5. What next?

Case Study
Pricing of a Corporate Bond

Case Study: Pricing a Corporate Bond I

- Trade Date: 7th of March 2018
- Maturity Date: 15 of May 2023
- Payment Frequency: Annual
- Day Count Convention: ACT/360, modified following business day
- Reset in Advance
- Current coupon rate
- Coupon formula

$$\text{CPN RATE} = \text{EURIBOR} + 20\ \text{BP; MIN CPN } 0.60\%.$$

- Consistently with market information (Libor rates, futures prices, swap rates, term structure of cap volatilities), provide the Gross Price, Accrued Interest and Clean Price.
- Estimate the CVA of the contract.

Data collection I

We have collected the following market information

- LIBOR rates

a ICE https://www.theice.com/marketdata/reports/170

- Swap rates

b ICE Source: https://www.theice.com/marketdata/reports/180

c Markit Calculator
https://www.markit.com/markit.jsp?jsppage=pv.jsp

- Volatility surface

d Source: no public available data

- Libor rates on the last reset date

e Source:
https://www.emmi-benchmarks.eu/euribor-org/euribor-rates.html

Data collection II

Table: Libor and Swap Rate Quotes

TERM	Rate	Market	TERM	Rate	Market
1 days	-0.44186	LIBOR	1 Year	-0.31	SWAP
1 Week	-0.422	LIBOR	2 Years	-0.122	SWAP
1 Month	-0.40657	LIBOR	3 Years	0.081	SWAP
2 Month	-0.39857	LIBOR	4 Years	0.284	SWAP
3 Month	-0.37929	LIBOR	5 Years	0.466	SWAP
6 Month	-0.33114	LIBOR	6 Years	0.625	SWAP
1 Year	-0.255	LIBOR	7 Years	0.765	SWAP
			8 Years	0.888	SWAP
			9 Years	0.998	SWAP
			10 Years	1.095	SWAP
			12 Years	1.256	SWAP
			15 Years	1.426	SWAP
			20 Years	1.568	SWAP
			25 Years	1.607	SWAP
			30 Years	1.611	SWAP

Data collection III

Table: LIBOR rates on the last reset date

	15/05/2017
1w	-0.380
2w	-0.373
1m	-0.374
2m	-0.341
3m	-0.330
6m	-0.251
9m	-0.179
12m	-0.127

Data collection IV

92) Actions | 93) Settings | Interest Rate Volatility Cube

EUR | EUR BVOL Cube (Default) | Mid | Date 03/02/18

95) Analyze Cube | 96) Market Data

11) Configuration | 12) Caps/Floors | 13) ATM Swaptions | 14) OTM Swaptions / SABR

Type: Normal Vol (OIS) | Source: BVOL | 16) Use This Contributor in Configuration

Table | Charts

Expiry	-0.50%	-0.25%	-0.13%	ATM	0.00%	0.13%	0.25%	0.50%	1.00%	1.50%	2.00%	2.50%
1Yr	15.50	11.15	14.14	10.84	17.93	21.72	25.42	32.53	45.90	58.55	70.70	82.50
18Mo	21.37	16.68	19.70	16.78	22.50	25.44	28.53	34.83	47.19	59.03	70.44	81.53
2Yr	24.00	19.54	22.20	21.49	24.17	26.07	28.27	33.53	45.24	57.00	68.53	79.85
3Yr	32.80	27.40	29.06	33.92	31.48	33.69	35.88	40.26	49.10	58.60	68.43	78.32
4Yr	38.12	33.72	35.10	43.32	37.36	39.54	41.68	45.80	53.69	61.92	70.58	79.49
5Yr	41.87	38.46	39.63	49.85	41.63	43.65	45.64	49.47	56.66	63.92	71.41	79.09
6Yr	45.38	42.59	43.51	54.51	45.21	46.97	48.75	52.17	58.53	64.76	71.08	77.53
7Yr	47.66	45.39	46.16	57.53	47.63	49.19	50.78	53.86	59.60	65.22	70.91	76.76
8Yr	50.06	48.07	48.66	59.77	49.88	51.23	52.61	55.33	60.41	65.33	70.27	75.33
9Yr	51.82	50.08	50.56	61.34	51.60	52.77	53.99	56.42	60.99	65.41	69.86	74.45
10Yr	53.06	51.55	51.96	62.54	52.87	53.92	55.02	57.23	61.42	65.48	69.60	73.86
12Yr	54.75	53.59	53.89	63.66	54.61	55.45	56.35	58.18	61.68	65.08	68.50	72.03
15Yr	55.88	55.13	55.36	64.06	55.91	56.56	57.26	58.72	61.55	64.33	67.16	70.09
20Yr	55.36	55.11	55.35	62.85	55.79	56.30	56.85	57.99	60.24	62.49	64.77	67.16
25Yr	54.50	54.55	54.81	61.51	55.21	55.65	56.13	57.11	59.08	61.07	63.11	65.26
30Yr	53.88	54.09	54.36	60.48	54.73	55.13	55.56	56.45	58.25	60.08	61.99	64.00

Coupon Profile and its decomposition

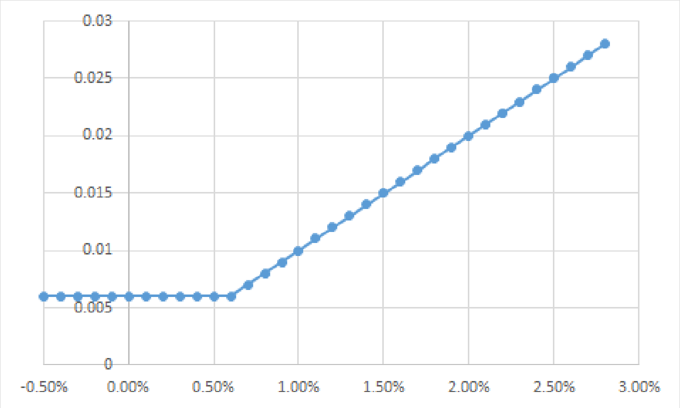


Figure: The coupon formula

The coupon can be decomposed as

$$0.6\% + caplet(K = 0.4\%)$$

The Pricing Procedure

- Build the discount curve
- Build the payment schedule
- Interpolate the market rates on the coupon payment dates
- Decompose the coupon in elementary components.
- Using the appropriate pricing model, price each component of the bond

Discount Curve

- It has been obtained using the usual bootstrapping procedure (we did not consider the weekends and holidays adjustments but this should be done).
- We have used the constant forward rate method to interpolate on the missing dates.
- Interpolate the market rates on the coupon payment dates
- Decompose the coupon in elementary components.
- Using the appropriate pricing model, price each component of the bond

TTM	DF	Tenor	Annuity	Fwd Rate	Fwd DF	Swap Rate	Fitting
0.002778	1.000012	0.002778					
0.019444	1.000082	0.019444					
0.083333	1.000339	0.083333					
0.166667	1.000665	0.166667					
0.25	1.000949	0.25					
0.5	1.001658	0.5					
1	1.00311	1	1.00311			-0.3100%	
2	1.002447	1	2.005556	0.0661%		-0.1220%	
3	0.997567	1	3.003124	0.4891%		0.0810%	
4	0.988663	1	3.991787	0.9006%		0.2840%	
5	0.976846	1	4.968633	1.2097%		0.4660%	
6	0.962928	1	5.931561	1.4454%		0.6250%	
7	0.947376	1	6.878937	1.6415%		0.7650%	
8	0.930651	1	7.809588	1.7972%		0.8880%	
9	0.912949	1	8.722537	1.9390%		0.9980%	
10	0.894691	1	9.617229	2.0407%		1.0950%	
11	0.875866	1	10.49309	2.1493%	0.978959	1.1830%	
12	0.857437	1	11.35053	2.1493%	0.978959	1.2560%	4.85E-11
13	0.838902	1	12.18943	2.2095%	0.978382	1.3216%	
14	0.820766	1	13.0102	2.2095%	0.978382	1.3776%	
15	0.803023	1	13.81322	2.2095%	0.978382	1.4260%	5.05E-10

Table: Bootstrapped Discount Curve

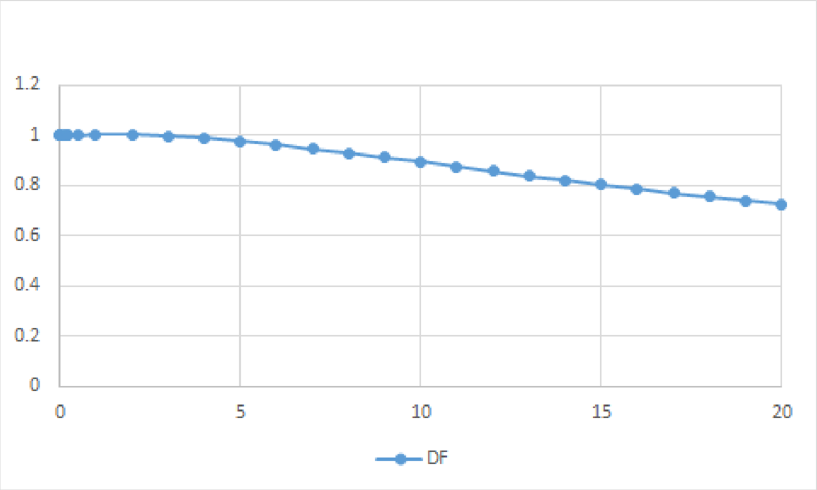


Figure: Bootstrapped Discount Curve

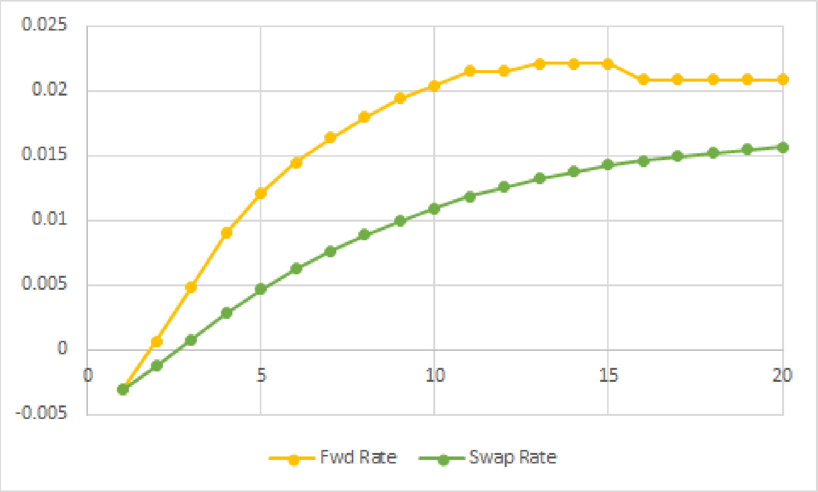


Figure: Swap Curve and bootstrapped Forward Curve

Coupon Number	Reset Date	Payment Starts	Payment Ends	Adjust for holidays	DAYS	CPN DAYS
			09 March 2018	09 March 2018	0	
1	15 May 2017	15 May 2017	15 May 2018	15 May 2018	67	365
2	15 May 2018	15 May 2018	15 May 2019	15 May 2019	432	365
3	15 May 2019	15 May 2019	15 May 2020	15 May 2020	798	366
4	15 May 2020	15 May 2020	15 May 2021	17 May 2021	1165	367
5	17 May 2021	17 May 2021	15 May 2022	16 May 2022	1529	364
6	16 May 2022	16 May 2022	15 May 2023	15 May 2023	1893	364

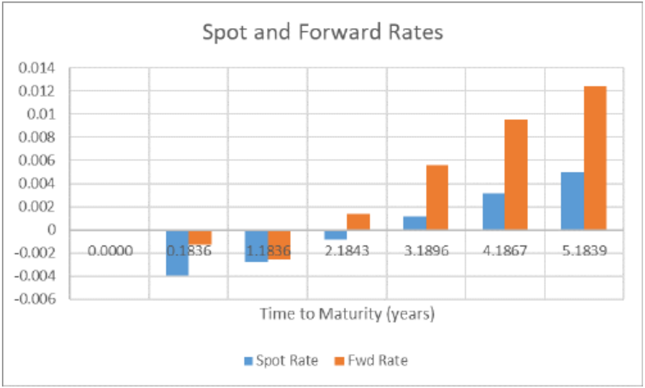
Table: Coupon Dates

Interpolated discount curve I

Table: (Linear) Interpolated Spot Rates and Discount Factors

DAYS	TTM	Int. Spot Rate	DF
67	0.183562	-0.00395	1.00072
432	1.183562	-0.00276	1.00327
798	2.184307	-0.00085	1.00185
1165	3.189596	0.001198	0.99619
1529	4.186747	0.003193	0.98672
1893	5.183934	0.004982	0.97451

Interpolated discount curve II



Interpolating Volatility

Table: Market Implied Volatility for different terms and strikes

	Strikes	
TERM	0.25%	0.50%
5	45.64	49.47
6	48.75	52.17

We (linearly) interpolate across strikes and maturities to obtain a volatility for the strike 0.4% and time to maturity of 5.19178 years equal to **48.48726** basis points.

Table: Pricing the bond components

		Fixed Component	Caplet Component						
Fwd Rate	AF	Fixed	Vol	Strike	TTM	d1	N(d1)	n(d1)	Fwd Premium
-0.127%	1.01	0.6083%		0.40%					0.0000%
-0.250%	1.01	0.6083%	0.48487%	0.40%	0.18356	-313.04%	0.09%	0.30%	0.0001%
0.139%	1.02	0.6100%	0.48487%	0.40%	1.18356	-49.42%	31.06%	35.31%	0.1070%
0.558%	1.02	0.6117%	0.48487%	0.40%	2.18431	22.05%	58.73%	38.94%	0.3790%
0.949%	1.01	0.6067%	0.48487%	0.40%	3.18960	63.36%	73.68%	32.64%	0.6945%
1.240%	1.01	0.6067%	0.48487%	0.40%	4.18675	84.62%	80.13%	27.89%	0.9600%

- The gray cell in the column forward rate refers to the rate (EURIBOR at reset date) to be used to compute the current coupon;
- Fixed component has been obtained by multiplying the Fixed Rate of 60 basis points by the accrual factor (AF)
- The caplet component has been obtained by computing the forward premium of the caplet according to the Bachelier model.

Table: Present Value of the coupons and Bond Pricing

Fixed	Fwd. Premium	Expected Coupon	DF	PV(Exp. Cpn)
0.6083%	0.0000%	0.6083%	1.00072	0.6088%
0.6083%	0.0001%	0.6084%	1.00327	0.6104%
0.6100%	0.1070%	0.7170%	1.00185	0.7184%
0.6117%	0.3790%	0.9907%	0.99619	0.9869%
0.6067%	0.6945%	1.3012%	0.98672	1.2839%
0.6067%	0.9600%	1.5666%	0.97451	1.5267%
			Sum	5.7351%

Valuation	
	PV
Coupons	5.7351%
Notional	97.4507%
Bond Gross Price	103.1857%
Current Coupon	0.6083%
Accrued Interest	0.4967%
Clean Price	102.6891%

Bond Sensitivity & Hedging I

- We examine the sensitivity of the bond to parallel shift of the term structure.
- We take the interpolated spot curve, we shift it up and down by 1 basis point and we recompute the discount curve and then we reprice the bond.

Table: Shifted Curve

DAYS	TTM	I Spot Rate	Shift	New Shifted Curve	DF
67	0.183562	-0.3948%	0.0100%	-0.3848%	1.00071
432	1.183562	-0.2759%	0.0100%	-0.2659%	1.00315
798	2.184307	-0.0847%	0.0100%	-0.0747%	1.00163
1165	3.189596	0.1198%	0.0100%	0.1298%	0.99587
1529	4.186747	0.3193%	0.0100%	0.3293%	0.98631
1893	5.183934	0.4982%	0.0100%	0.5082%	0.97400

- Given the new shifted curve we reprice the bond.
- We also consider a downward shift in the curve

Bond Sensitivity & Hedging II

Table: Sensitivity is computed as (GBPUp-GBPDown)/2

Shift	Gross Bond Price
+1BP	103.158%
0	103.186%
-1BP	103.214%
Sensitivity	**-0.028%**

- In order to hedge the position we consider a 5 years swap with semi-annual cash flows on the two legs.
- The swap is priced in the sheet Swap Pricing and has a Fixed Rate equal to 0.5167%
- Keeping constant the fixed rate, we reprice the swap in the two states (up and down shift).

Bond Sensitivity & Hedging III

- We obtain the following Table

Table: Repricing the bond and the swap given term structure shifts

Shift	Gross Bond Price	Swap
+1BP	103.158%	0.0514%
0	103.186%	0.0000%
-1BP	103.214%	-0.0515%
Sensitivity	-0.0282%	0.0515%

Bond Sensitivity & Hedging IV

- Therefore, if we aim to hedge our portfolio we can add a number n of swaps, so that
$$\pi(t) = GBP(t) + nSwap(t)$$
where, by construction, $Swap(t) = 0$.
- However the sensitivity is
$$\Delta\pi(t) = (-0.0282\%) + n \times (0.0515\%).$$
- So in order to hedge our portfolio, we need a number $\hat{n}$ of swpas
$$\hat{n} = -\frac{-0.0282\%}{0.0515\%} = 0.5473.$$

Pricing a defaultable bond

- We have assumed that the issuer does not bear any default risk.
  - We need to correct the market value for early default.
  - We can use the general formula for pricing a risky note:

$$V_D(t) = \underbrace{\sum_{i=1}^{n} Q(t, T_i) \times P(t, T_i) \times \alpha_{T_{i-1},T_i} \times \mathbb{E}_t(CF(T_i)) \times N}_{\text{expected coupon payment if survives up to } T_i}$$
$$+ \underbrace{Q(t, T_n) \times P(t, T_n) \times N}_{\text{notional payment if survives up to } T_n}$$
$$+ N \times R \times \sum_{i=1}^{n} \underbrace{(Q(t, T_{i-1}) - Q(t, T_i))}_{prob.\ of\ default\ in\ \left(T_{i-1}, T_i\right]} \times P(t, T_i), \quad (1)$$

  where $CF(T_i)$ is the random annual coupon due at time T_i and $\alpha_{T_{i-1},T_i} \times \mathbb{E}_t(CF(T_i))$ is the so called forward expected cash flow.
  - In our case the forward premium is related to

$$\alpha_{T_{i-1},T_i} \times (0.6\% + FwdCapletPremium)$$

  where Fwd Caplet Premium is estimated using the Bachelier model.

The Risky Component I

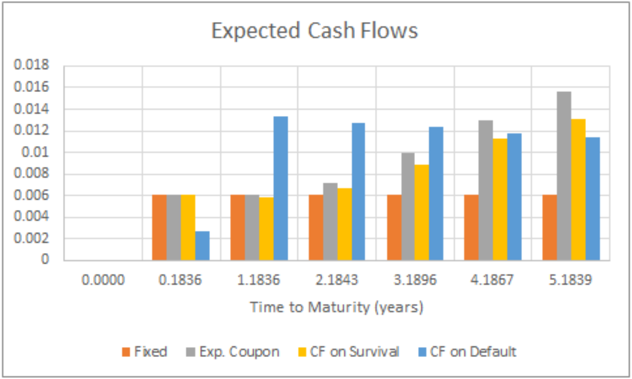
Table: Computing the Risky value of the bond

Risky Valuation					
Survival Probability	Recovery	CF on Survival	PV(CF) On survival	CF on Default	PV on Default
1					
99.320%	40%	0.604%	0.605%	0.272%	0.272%
95.978%	40%	0.584%	0.586%	1.337%	1.341%
92.802%	40%	0.665%	0.666%	1.270%	1.272%
89.722%	40%	0.888%	0.885%	1.232%	1.227%
86.767%	40%	1.128%	1.113%	1.182%	1.166%
83.908%	40%	1.314%	1.280%	1.144%	1.115%

The Risky Component II

Table: Computing the Bond CVA

Risky Adjusted Valuation	
	PV
Coupons on Survival	5.1357%
Notional on Survival	81.7727%
CF on Default	6.3945%
Bond Gross Price	93.3029%
CVA	9.8856%


Expected Cash Flows
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0.0000
0.1836
1.1836
2.1843
3.1896
4.1867
5.1839
Time to Maturity (years)
Fixed
Exp. Coupon
CF on Survival
CF on Default

What next?

- The reference rate of the corporate bond is a 1yr LIBOR rate.
  - Euro market quotations are relative to the 6m LIBOR rate.
  - We should transform this volatility into the 1yr volatility.
  - Ideas?
- We have considered how to hedge against parallel shift
  - How to deal with non-parallel shifts?
  - Ideas?
- How do you hedge against deterioration in the credit risk of the issuer?
  - Ideas?

Pricing Models for Swaptions: Black, Bachelier and Displaced

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MSc Mathematical Finance & Trading
MSc Quantitative Finance

SMM269 Fixed Income - Academic Year 2019-20

Main References:

Useful Readings

- Brigo Damiano and Fabio Mercurio, Interest Rate Models: Theory and Practice, Springer Finance 2001.
- Pietro Veronesi. Fixed Income Securities. **Chapter 20.**
- Bruce Tuckman, Angel Serrat. Fixed Income Securities: Tools for Today's Markets, 3rd Edition **Chapter 18**.
- Interest rate derivatives in the negative-rate environment Pricing with a shift, Deloitte, Feb 2016.
- Options valuation strained by quantitative easing, Sungard.

Excel Files

- FI_BlackModel&co.xlsm

Outline

Swaption Payoff

- A European payer (receiver) swaption is an option giving the right (and no obligation) to enter a payer (receiver) IRS at the swaption maturity (a payer IRS pays a fixed rate and receive a floating rate).
- The payer swaption payoff at time T is

Payer swaption CALL

$$N \times [S(T, T_0, T_n) - K]^+ \times \sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} \times P(T, T_i),$$

where

$$S(T, T_0, T_n) = \frac{P(T, T_0) - P(T, T_n)}{\sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} \times P(T, T_i)}.$$

- The receiver swaption payoff is

Put

$$N \times [K - S(T, T_0, T_n)]^+ \times \sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} \times P(T, T_i).$$

- For pricing these swaptions, we need a model for the forward swap rate.

The Black model for swaption

Black formula for swaptions

If the forward swap rate has dynamics under the pricing measure given by

$$dS\left(t, T_0, T_n\right) = \sigma_S S\left(t, T_0, T_n\right) dW^{pricing}(t), t \leq T_0,$$

So given

the values at date t of payer and receiver swaptions are

$$\begin{aligned}
\text{payer(t)} &= \sum_{i=1}^{n} \alpha_{i-1,i} P\left(t, T_i\right) \times \left(S\left(t, T_0, T_n\right) \times \mathcal{N}\left(d_1\right) - K \times \mathcal{N}\left(d_2\right)\right) \times N, \\
\text{rec.(t)} &= \sum_{i=1}^{n} \alpha_{i-1,i} P\left(t, T_i\right) \times \left(K \times \mathcal{N}\left(-d_2\right) - S\left(t, T_0, T_n\right) \times \mathcal{N}\left(-d_1\right)\right) \times N,
\end{aligned}$$

FWD premium

where:

$$d_{1,2} = \frac{\ln\left(\frac{S(t,T,T_n)}{K}\right) \pm \frac{1}{2}\sigma_S^2\left(T-t\right)}{\sigma_S\sqrt{T-t}},$$

and σ_S is the percentage volatility of the forward swap rate.

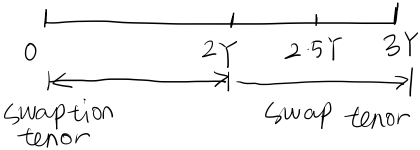
Example (Pricing a swaption)

We have a 2x1 swaption. The swap is spot starting with semi-annual payments and the reference rate is the 6m Libor rate. The strike is 10%. The percentage volatility of the relevant forward swap rate is 20%. The term structure of discount factors is given in the following Table

Term	0.5	1	1.5	2	2.5	3
DF	0.99	0.98	0.95	0.9045	0.8825	0.7866

Price the swaption.

$$S(0) = \frac{0.9045 - 0.7866}{0.5 \times (0.8825 + 0.7866)}$$



Swaption Value:

$$= \left(S_0 N(d_1) - 0.1 N(d_2)\right) \text{Annuity}$$

Example (2. Determining the swap payment dates)

- The swaption expires in 2 years.
- It gives the right to enter into a spot starting swap with 1 year tenor and semi-annual payments.
- The relevant payment dates of the swap are shown in the following scheme

Term	0.5	1	1.5	2	2.5	3
DF	0.99	0.98	0.95	0.9045	0.8825	0.7866
Reset				1st Reset	2nd Reset	
Payments					1st Payment	2nd Payment

Example (3. Applying the Black formula)

- Given the payment dates of the underlying swap, we can compute the forward swap rate. It is given by

$$S(0,2,3) = \frac{0.9045 - 0.7866}{(2.5-2)\times 0.8825 + (3-2.5)\times 0.7866} = \frac{0.1182}{0.8346} = 14.16\%.$$

- The Annuity appears in the denominator of the forward swap rate and equals 0.8346.
- The fixed swap rate (swaption strike) is 10%.
- In addition

$$d_1 = \frac{\ln\left(\frac{0.1416}{0.1}\right) + \frac{1}{2}\times 0.2^2 \times 2}{\sqrt{0.2^2\times 2}} = 1.3723,$$

$$d_2 = d_1 - 0.2\times\sqrt{2} = 1.0894.$$

- Therefore

$$swptn(0) = 0.8346 \times (14.16\%\mathcal{N}(1.3723) - 10\%\mathcal{N}(1.0894)) = 0.0362.$$

$$S(T) = \frac{P(T,T_0) - P(T,T_N)}{\text{Annuity}} = \frac{\sum F(T,T_{i-1},T_i)\,\alpha_{i-1,i}\,P(T,T_i)}{\sum \alpha_{i-1,i}\,P(T,T_i)}$$

Inconsistency of the Black formula for caplets and swaptions I

- In pricing cap with the Black formula, we assume that forward interest rates $F\left(T_{i,}T_i, T_{i+1}\right)$ are lognormally distributed.
- In pricing swaptions, the Black model assumes that swap rate $S\left(T, T_0, T_n\right)$ is lognormally distributed.
- But we have seen that $S\left(T, T_0, T_n\right)$ is a weighted average of simple forward rates:

$$S\left(T, T_0, T_n\right) = \sum_{i=1}^{n} w_{i-1} F\left(T, T_{i-1}, T_i\right)$$

with weights:

$$w_{i-1} = \frac{P\left(T, T_i\right)\alpha_{i-1,i}}{\sum_{i=1}^{n} P\left(T, T_i\right)\alpha_{i-1,i}}.$$

Inconsistency of the Black formula for caplets and swaptions II

- If we assume that $F(T, T_{i-1}, T_i)$ are lognormal, we cannot assume that $S(t, T_0, T_n)$ is lognormal as well: indeed, the sum of lognormals is not lognormal.
- So the two models for pricing caps and swaptions are logically inconsistent.
- Nevertheless the financial markets price both caps and swaptions using Black models.
- This incompatibility is mostly theoretical: in practice the distribution of the forward swap rate is almost lognormal.

The surface of swaption implied volatilities

Implied volatility matrix for swaption

- As the market quote flat volatilities for caps, in a similar manner swaption prices are usually quoted as the volatility σ_S.
- So the market quotes a volatility for a given time to maturity of the swaption, $T - t$, and for a given tenor of the underlying (spot starting) swap, $T_n - T$.

Market quotations of swaption implied volatilities

<HELP> for explanation. DL18 Equity SWYV
Enter 1 <Go> to save setting

Euro Volatility
Swaption Implied

(Bid/Ask/Mid) Page 2/3

Option Expiry	Term 1 yr	2 yr	3 yr	4 yr	5 yr	7 yr	10 yr
1 mo	28.34 C	33.30 C	31.50 C	28.22 C	26.73 C	23.10 C	19.07 C
3 mo	30.05 C	31.07 C	29.06 C	26.76 C	25.28 C	21.55 C	18.22 C
6 mo	30.47 C	30.14 C	27.60 C	25.35 C	23.29 C	20.19 C	17.31 C
9 mo							
1 yr	29.97 C	27.22 C	24.36 C	22.30 C	20.72 C	18.22 C	16.04 C
2 yr	24.92 C	22.39 C	20.36 C	18.88 C	17.64 C	15.98 C	14.47 C
3 yr	21.00 C	19.30 C	17.46 C	16.70 C	15.53 C	14.49 C	13.60 C
4 yr	17.55 C	16.41 C	15.57 C	14.78 C	14.05 C	13.39 C	12.70 C
5 yr	15.80 C	14.92 C	14.19 C	13.45 C	12.99 C	12.54 C	12.02 C
7 yr	13.40 C	12.80 C	12.40 C	12.00 C	11.80 C	11.60 C	11.20 C
10 yr	12.00 C	11.50 C	11.20 C	11.00 C	10.90 C	10.75 C	10.67 C

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Market quotations of swaption implied volatilities

<HELP> for explanation. DL18 Equity **SWYV**

Euro Volatility
Swaption Implied

B (Bid/Ask/Mid) Page 3/3

Option Expiry	7 yr	10 yr	12 yr	15 yr	20 yr	25 yr	30 yr
1 mo	23.10 C	19.07 C		16.70 C	15.50 C	14.90 C	14.60 C
3 mo	21.51 C	18.04 C		16.10 C	15.00 C	14.50 C	14.10 C
6 mo	20.08 C	17.23 C		15.40 C	14.50 C	13.90 C	13.60 C
9 mo							
1 yr	18.26 C	15.87 C		14.50 C	13.60 C	13.20 C	13.00 C
2 yr	16.05 C	14.24 C		13.40 C	12.60 C	12.30 C	12.20 C
3 yr	14.49 C	13.60 C		12.70 C	12.10 C	11.80 C	11.70 C
4 yr	13.39 C	12.70 C		12.10 C	11.60 C	11.40 C	11.30 C
5 yr	12.54 C	12.02 C		11.50 C	11.10 C	11.00 C	10.90 C
7 yr	11.60 C	11.20 C		10.70 C	10.40 C	10.40 C	10.40 C
10 yr	10.75 C	10.67 C		10.20 C	10.00 C	9.90 C	10.00 C

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Displaced Black for swaptions

Displaced Black formula for swaptions

If the forward swap rate has dynamics under the pricing measure given by

$$dS(t, T_0, T_n) = \sigma_S (S(t, T_0, T_n) + \delta) \, dW^{pricing}(t), t \leq T_0$$

the values at date t of payer and receiver swaption are

$$\text{payer(t)} =$$
$$\sum_{i=1}^{n} \alpha_{i-1,i} P(t, T_i) \times ((S(t, T_0, T_n) + \delta) \times \mathcal{N}(d_1) - (K + \delta) \times \mathcal{N}(d_2)) \times N,$$
$$\text{receiver(t)} =$$
$$\sum_{i=1}^{n} \alpha_{i-1,i} P(t, T_i) \times ((K + \delta) \times \mathcal{N}(-d_2) - (S(t, T_0, T_n) + \delta) \times \mathcal{N}(-d_1)) \times N$$

where:

$$d_{1,2} = \frac{\ln\left(\frac{S(t,T,T_n)+\delta}{K+\delta}\right) \pm \frac{1}{2}\sigma_S^2 (T - t)}{\sigma_S \sqrt{T - t}},$$

and σ_S is the percentage volatility of the forward swap rate.

Question

Price a two year swaption to enter into a new 1 yr swap (spot starting) with semi-annual payments: $T = 2$ (option maturity), $T_1 = 2.5$ (first swap date), $T_2 = 3$ (second swap date). The swaption strike is 0. The volatility is 20%. The following Table provides information on the discount factors.

	Maturity	Rate (c.c.)	$P(t, T_i)$
T	2.00	-0.05%	1.0010
T_1	2.50	-0.01%	1.0003
T_2	3.00	0.02%	0.9994

Example (Pricing a swaption with the displaced Black Model)

Table: Market Rates

	Maturity	Rate (c.c.)	$P(t, T_i)$
T	2.00	-0.05%	1.0010
T_1	2.50	-0.01%	1.0003
T_2	3.00	0.02%	0.9994

Table: Pricing the swaption

K	0.00%
$T - t$	2
σ	20.00%
δ	1.00%
Annuity	0.9998
$S(t, T, T_n)$	0.160%
d_1	0.6663
d_2	0.3835
Swaption	**0.0022**

$A = 0.5 \times (1.0003 + 0.9994)$

$= 0.9998$

$S = \frac{1.001 - 0.9994}{0.9998}$

$d_{1,2} = \frac{\ln\left(\frac{0.16\% + 1\%}{0\% + 1\%}\right) \pm \frac{1}{2}(0.2)^2 \times 2}{\sqrt{0.2^2 \times 2}}$

$= 0.9998 \times (0.160\% N(d_1) - 0 \times N(d_2))$

$= 0.0022$

The Bachelier Formula for swaptions

Bachelier formula for swaptions

If the forward swap rate has dynamics under the pricing measure given by

$$\begin{cases} dS\left(t, T_0, T_n\right) = \sigma_S dW^{pricing}(t), t \leq T_0 \\ S(t) \text{ given} \end{cases}$$

S is a martingale

the values at date t of payer and receiver swaption are

$$\text{payer(t)} = \sum_{i=1}^{n} \alpha_{i-1,i} P\left(t, T_i\right) \times \left(\left(S\left(t, T_0, T_n\right) - K\right) \mathcal{N}\left(d_1\right) + \sqrt{V(t,T)} n\left(d_1\right)\right),$$

$$\text{receiver(t)} = \sum_{i=1}^{n} \alpha_{i-1,i} P\left(t, T_i\right) \times \left(\left(K - S\left(t, T_0, T_n\right)\right) \mathcal{N}\left(-d_1\right) + \sqrt{V(t,T)} n\left(d_1\right)\right),$$

where $V(t,T) = \sigma_S^2 \times (T - t)$, and:

$$d_1 = \frac{S\left(t, T_0, T_n\right) - K}{\sqrt{V(t,T)}},\ \mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz, n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

and σ_S is the normal volatility of the forward swap rate.

Example (Pricing a swaption using Bachelier formula)

We have to price a 6mx6m swaption. The swap has quarterly payments

- The 6m, 9m and 12m LIBOR rates are -0.32057% -0.21214% -0.08329%.
- Corresponding discount factors are 1.00161, 1.00159 and 1.00083.
- The swaption strike is 0%.
- The (constant) absolute volatility is 2.50%.

Example ((...) Pricing a swaption using Bachelier formula)

Therefore

- The swaption expires in 6 months.
- The swap resets in 6m and 9m and pays in 9m and 12m.

Time	0	0.5	0.75	1
DF		1.00161	1.00159	1.00083
Reset				
Payment				

- The forward swap rate is

$$S = \frac{1.00161 - 1.00083}{0.25\,(1.00159 + 1.00083)} = \frac{0.00077}{0.5006} = 0.1542\%.$$

- In addition

$$d_1 = \frac{0.1542\% - 0}{0.025 \times \sqrt{0.5}} = 0.0872, \mathcal{N}(d_1) = 0.5348, n(d_1) = 0.3974.$$

- Therefore the swaption price is

$$0.5006 \times \left((0.15421\% - 0) \times 0.5348 + 0.025 \times \sqrt{0.5} \times 0.3974\right) = \mathbf{0.0039}.$$

Homework

- Let us suppose to have a flat LIBOR term structure at 0.5%.
- Price a 3x2 swaption on the 3m LIBOR (payments are quarterly) given that the strike price is 0.25% and the percentage volatility of the forward swap rate is at 20%.
- Price a 3x2 swaption, using the Bachelier model, on the 3m LIBOR (payments are quarterly) given that the strike price is 0.25% and the absolute volatility of the forward swap rate is at 100 bp.
- Price a 3x2 swaption, using the displaced Black model, on the 3m LIBOR (payments are quarterly) given that the strike price is -1% and the percentage volatility of the forward swap rate is at 20% and the shift coefficient is -1%.

Conclusions

We have presented

- The swaption payoff
- The most popular pricing models
  - Black
  - Displaced Black
  - Bachelier

Appendix

Swaption volatilities and forward rate volatilities

- The relationship between forward rates and forward swap rate allow us to find `absolute` swaption volatilities from `absolute` forward rate volatilities.
- Indeed, freezing the terms w_i, we have:

$$
\begin{aligned}
\mathbb{V}ar\left(S\right) &= \mathbb{V}ar\left(\sum_{i=1}^{n} w_{i-1}F\left(T, T_{i-1}, T_i\right)\right) \\
&= \sum_{i,j=1}^{n} w_{i-1}w_{j-1}\sigma_{i-1,j-1}^2
\end{aligned}
$$

where $\sigma_{i,j}^2$ is the covariance between $F\left(T, T_{i-1}, T_i\right)$ and $F\left(T, T_{j-1}, T_j\right)$:

$$
\sigma_{i-1,j-1}^2 = \mathbb{C}ov\left(F\left(T, T_{i-1}, T_i\right); F\left(T, T_{j-1}, T_j\right)\right).
$$

Swaption volatilities and forward rate volatilities (ctd)

- In practice, this approach is adequate to provide an indication where the swaption volatility should be, but the swaption market has its own characteristics as distinct from the cap market.
- Relying on this relationship for pricing, and more for risk management, would introduce considerable basis risk.

Change of Numeraire and Pricing of Interest Rate Derivatives

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MSc Mathematical Finance & Trading
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SMM269 Fixed Income

Academic Year 2019-20

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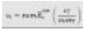
Useful Web Resources

- `https://sites.google.com/uniupo.it/stochasticcalculus/home?authuser=1`

Outline I



Numeraire and Martingale

Pricing problem

- We have to price an instrument with cash-flow
  - for a zcb:
$$c(T) = 1;$$
  - for a caplet:
$$c(T_2) = (F(T_1, T_1, T_2) - K)^+ \times \alpha_{T_1, T_2};$$
  - for a swaption:
$$c(T) = (S(T, T_0, T_n) - K)^+ \times \sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} P(T, T_i).$$
  - for a bond option:
$$c(T) = \left(\sum_{i=1}^{n} c_i \alpha_{T_{i-1}, T_i} P(T, T_i) - K \right)^+.$$
  - and much more: stock options when rates are stochastic, exchange options, etc.
- The pricing problem can be solved introducing the concept of numeraire and martingale process.

Important concepts

In the following, we will use

- Numeraire asset;
- Relative Price;
- Martingale process;
- No-Arbitrage.

Numeraire Asset

Fact (Numeraire)

- *A numeraire is a particular asset that can be used to price relative to. Numeraire means a unit of measurement.*
- *A numeraire must have strictly positive value and must be self-financing.*

- Examples of numeraire are:
  - the money market account;
  - a non dividend paying asset;
  - a zcb maturing at a suitable date;
  - a constant maturity coupon bond.

Relative Price

Fact (Relative Price)

- *The ratio of one price to another is the value of the first (numerator) asset when we are using the second (denominator) asset as the numeraire.*
- *If $num(t)$ is the value at time t of the numeraire, then for an asset with value $v(t)$ its value relative $RP(t)$ to the numeraire is:*

$$RP(t) = \frac{v(t)}{num(t)}.$$

Martingale

Martingale

A stochastic process $(X(t))_{t\geq 0}$ is a martingale with respect to a measure $\mathbb{Q}$ and a filtration $\mathcal{F}_t$ if its expected future value equals its current value

$$\mathbb{E}_t^{\mathbb{Q}}(X(T)) \equiv \mathbb{E}^{\mathbb{Q}}(X(T)\,|\mathcal{F}_t) = X(t), \forall T \geq t,$$

where $\mathbb{E}_t^{\mathbb{Q}}$ is the t-conditional expectation with respect to the probability measure $\mathbb{Q}$.

In the following, we will use the shorthand notation

$$\mathbb{E}_t^{\mathbb{Q}}(X(T))$$

to denote conditional expectations.

Example

- Consider a random process X that lives one period, starts at 100 and dies in 80 or 120.
- We can associate to it different probability measures, $\mathbb{P}$ =(0.4; 0.6)and $\mathbb{Q}$ = (0.5; 0.5) say.
- X is a martingale wrt $\mathbb{Q}$ and not wrt $\mathbb{P}$:

Not a Martingale Process	A Martingale Process
$\begin{array}{ccc} & & \mathbb{P} \\ & 120 & 0.4 \\ \nearrow & & \\ 100 & & \\ \searrow & & \\ & 80 & 0.6 \end{array}$	$\begin{array}{ccc} & & \mathbb{Q} \\ & 120 & 0.5 \\ \nearrow & & \\ 100 & & \\ \searrow & & \\ & 80 & 0.5 \end{array}$
$100 \neq 0.4 \times 120 + 0.6 \times 80$	$100 = 0.5 \times 120 + 0.5 \times 80$

- In the example $\mathbb{Q}$ is a probability measure that is equivalent to $\mathbb{P}$ (i.e. for any set $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ if and only if $\mathbb{P}(A) = 0$).

Example

- Two equivalent measures and two not-equivalent measures.

Equivalent Measures $\mathbb{P} \sim \mathbb{Q}$

			$\mathbb{P}$				$\mathbb{Q}$
		120	0.4			120	0.3
	$\nearrow$				$\nearrow$		
100				100			
	$\searrow$				$\searrow$		
		80	0.6			80	0.7

Not Equivalent Measures $\mathbb{P} \nsim \mathbb{Q}$

			$\mathbb{P}$				$\mathbb{Q}$
		$\boxed{120}$	0.4			$\boxed{130}$	0.3
	$\nearrow$				$\nearrow$		
100				100			
	$\searrow$				$\searrow$		
		80	0.6			80	0.7

Fundamental Theorem of Asset Pricing: Relative prices are Martingales

Problem: How to use a numeraire for pricing?

Main result (Harrison and Kreps)

If there are no arbitrage opportunities, then relative prices are martingales under risk-adjusted probabilities, i.e., given a numeraire num_t there exists a probability measure $\mathbb{Q}$ such that the ratio of any other asset price v_t to the numeraire is a martingale, i.e. for all $t < T$

$$\frac{v(t)}{num(t)} = \mathbb{E}_t^{\mathbb{Q}}\left(\frac{v(T)}{num(T)}\right).$$

or equivalently $v(t) = num(t) \times \mathbb{E}_t^{\mathbb{Q}}\left(\frac{v(T)}{num(T)}\right)$. To emphasize that the numeraire can be any non-dividend-paying asset, we can write the no-arbitrage formula in the form:

$$v(t) = num(t) \times \mathbb{E}_t^{num}\left(\frac{v(T)}{num(T)}\right).$$

Remarks

- Different numeraires lead to different probability measures and hence to different expectations.
- The absence of arbitrage does not depend on the choice of the probability measure an on the choice of the numeraire.
- If we change probability measure (and numeraire) the no-arbitrage condition must still be satisfied.

Change of numeraire and change of measure I

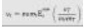
Consider the payoff $G(T)$. We can price it using different numeraire and different probability measures.

- Let $\mathbb{Q}^N$ be the equivalent martingale measure with respect to the numeraire $N(t)$. Therefore

$$G(t) = N(t)\mathbb{E}_t^N\left[\frac{G(T)}{N(T)}\right]. \tag{1}$$

- Let us consider a new numeraire $M(t)$. How do we find the corresponding probability measure?
- We re-arrange 1 and we write

$$G(t) = M(t)\mathbb{E}_t^N\left[\frac{M(T)}{N(T)}\frac{N(t)}{M(t)}\frac{G(T)}{M(T)}\right]. \tag{2}$$



Change of numeraire and change of measure II

- Let us define the so-called Radon-Nikodym derivative

$$\psi(T) = \frac{M(T)}{N(T)} \frac{N(t)}{M(t)}. \tag{3}$$

- Given that an expectation is an integral, formula (2) amounts to write

$$\begin{aligned} G(t) &== M(t) \int \frac{G(T)}{M(T)} \frac{M(T)}{N(T)} \frac{N(t)}{M(t)} dQ^N(T) \\ &= M(t) \int \frac{G(T)}{M(T)} \psi(T) dQ^N(T) \end{aligned}$$

- Let us define

$$d\mathbb{Q}^M = \psi(T) d\mathbb{Q}^N(T).$$

Change of numeraire and change of measure III

- We observe that $d\mathbb{Q}^M$ is a probability measure:

a. it is positive (product of positive quantities)

b. it integrates to 1

$$
\begin{aligned}
\int d\mathbb{Q}^M &= \int \psi(T) d\mathbb{Q}^N(T) \\
&= \mathbb{E}_t^N \left[\frac{N(t)}{M(t)} \frac{M(T)}{N(T)} \right] \\
&= \frac{N(t)}{M(t)} \mathbb{E}_t^N \left[\frac{M(T)}{N(T)} \right] \\
&= \frac{N(t)}{M(t)} \frac{M(t)}{N(t)} \\
&= 1.
\end{aligned}
$$

where we have exploited the fact that $M(T)/N(T)$ is a martingale if N is the numeraire.

Change of numeraire and change of measure IV

- Therefore (**??**) becomes

$$
\begin{aligned}
G(t) &= M(t) \int \frac{G(T)}{M(T)} \psi(T) d\mathbb{Q}^N(T) \\
&= M(t) \int \frac{G(T)}{M(T)} d\mathbb{Q}^M \\
&= M(t) \mathbb{E}_t^M \left[\frac{G(T)}{M(T)} \right].
\end{aligned}
$$

Take Away 1

1. Given a payoff $G(T)$, we can determine its price selecting a numeraire $N(t)$ and a corresponding probability measure $d\mathbb{Q}^M$. Therefore
$$G(t) = N(t)\mathbb{E}_t^N\left[\frac{G(T)}{N(T)}\right].$$
2. However, the price of the payoff does not change if we change numeraire and probability measure
$$G(t) = M(t)\mathbb{E}_t^M\left[\frac{G(T)}{M(T)}\right].$$
3. We can move from the probability measure $d\mathbb{Q}^N$ to the $d\mathbb{Q}^M$ (and viceversa) using the Radon-Nykodym
$$\psi(T) = \frac{d\mathbb{Q}^M}{d\mathbb{Q}^N} = \frac{M(T)}{N(T)}\frac{N(t)}{M(t)}, \tag{4}$$
and then
$$d\mathbb{Q}^M = \psi(T)d\mathbb{Q}^N(T),$$
or viceversa
$$d\mathbb{Q}^N = \frac{1}{\psi(T)}d\mathbb{Q}^M(T).$$

Take Away 2

How do we use these results?

1. Depending on the payoff, sometimes it is useful to price using the numeraire N (and $d\mathbb{Q}^N$) and sometimes using the numeraire M (and $d\mathbb{Q}^M$).
   - It will be a question of *convenient choice of the numeraire*.
2. Sometimes, we have a model specified under the probability measure $d\mathbb{Q}^N$ and we need to rewrite the model under the new the probability measure $d\mathbb{Q}^M$.
   - We will change probability measure (and we change numeraire) by using the Radon-Nykodim derivative.

Take Away 3: Risk Neutral Measure & MMA

- The `money market account` (MMA) gives us the T bank account value given that we have posted $B(t)$ at time t

$$B(T) = B(t) \times \exp\left(\int_t^T r(s)ds\right),$$

and $r(s)$ is the instantaneous return on the bank account.

- $B(t)$ can be considered a numeraire asset: it is traded and it has positive value.
- If we take the MMA as numeraire, the corresponding probability martingale measure is called "`risk neutral`" (RN) and we use the notation $\tilde{\mathbb{E}}_t(.)$ to denote conditional expectation under this measure.

Take Away 4: Pricing using the Risk Neutral Measure & MMA

Derivatives with payoff $v(T)$ can be priced using the MMA and the RN measure according to

$$\frac{v(t)}{B(t)} = \tilde{\mathbb{E}}_t\left(\frac{v(T)}{B(T)}\right)$$

and then we get the familiar pricing formula

$$v(t) = \tilde{\mathbb{E}}_t\left(\frac{v(T)}{\exp\left(\int_t^T r(s)ds\right)}\right) = \tilde{\mathbb{E}}_t\left(\exp\left(-\int_t^T r(s)ds\right)v(T)\right).$$

- The formula above suggests the statement *the derivative price is the expected value, using the RN measure, of the discounted payoff*.
- Moreover, if interest rates are constant $B(T) = B(t)e^{r(T-t)}$ and

$$v(t) = e^{-r\times(T-t)}\tilde{\mathbb{E}}_t\left(v(T)\right).$$

- But pricing is more general than computing discounted expectations under the RN measure. Using a different numeraire the pricing formula will be different.

Example :
Pricing a zero-coupon bond using the risk neutral measure

Example (1. ZCB pricing using the RN measure)

- Let us consider a pure discount bond, whose payoff is: $v(T) = 1$.
- Let us use the money market account as numeraire.
- The current price $P(t, T)$ of the zcb is therefore:

$$P(t, T) = \tilde{\mathbb{E}}_t \left(\exp\left(-\int_t^T r(s) ds \right) \times 1 \right).$$

- This is the approach used in short rate models, where a risk-neutral dynamics is assigned to the short rate

$$dr(t) = \mu(r, t)dt + \sigma(r, t)d\tilde{W}(t).$$

- Then the zcb price can be found if we know the moment generating function (MGF) of the time integral

$$I(t, T) = \int_t^T r(s) ds,$$

i.e. if we are able to compute $\tilde{\mathbb{E}}_t (\exp(-I(t, T)))$.

Example ((...ctd) Pricing zcb using the RN measure)

- The procedure gives closed form expression for zcb prices depending on the choice of the drift and diffusion coefficient.

Table: Legend: HW: Hull & White, CIR: Cox-Ingersoll-Ross, MR-LN: mean-reverting lognormal.

Model	Drift	Diffusion	Distr of $r(s)$	Distr of $I(t, T)$	MGF of $I(t,$
Merton	μ	σ	Gaussian	Gaussian	✓
Ho & Lee	$\mu(t)$	σ	Gaussian	Gaussian	✓
Vasicek	$\alpha(\mu - r(t))$	σ	Gaussian	Gaussian	✓
HW	$\alpha(\mu(t) - r(t))$	σ	Gaussian	Gaussian	✓
CIR	$\alpha(\mu - r(t))$	$\sigma\sqrt{r(t)}$	Non-Cent. χ^2	Unknown	✓
Dothan	$\mu r(t)$	$\sigma \times r(t)$	Lognormal	Unknown	✗
MR-LN	$\alpha(\mu - r(t))$	$\sigma \times r(t)$	Unknown	Unknown	✗

- The above Table shows that only for some model (Merton, HL, Vasicek, HW and CIR) the zcb price is available in closed form. Therefore, their popularity at least in the academic literature.

Example ((...ctd) Zcb pricing using the Merton model)

- The sde

$$dr(t) = \mu dt + \sigma d\tilde{W}(s).$$

- The model
  $r(s) = r(t) + \mu(s-t) + \sigma \int_t^s d\tilde{W}(u) \sim \mathcal{N}\left(r(t) + \mu(s-t), \sigma^2(s-t)\right)$
- Integrate

$$\int_t^T r(s)ds = r(t)(T-t) + \mu\frac{(T-t)^2}{2} + \sigma\int_t^T\int_t^s d\tilde{W}(u)ds$$

- Therefore $\int_t^T r(s)ds \sim \mathcal{N}\left(M(t,T), V(t,T)\right)$, where

$$M(t,T) = r(t)(T-t) + \mu\frac{(T-t)^2}{2}, \quad V(t,T) = +\frac{\sigma^2}{3}(T-t)^3.$$

- It follows that

$$P(t,T) = \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right) = e^{-M(t,T)+\frac{1}{2}V(t,T)}.$$

Example 2: Pricing a cash-or-nothing option using the risk neutral measure

Example (2. Pricing a cash-or-nothing option)

- A cash or nothing pays $1_{S(T)>K}$.
- Let us use the risk-neutral measure and the money-market account.
- The option price is:

$$v(t) = B(t) \times \tilde{\mathbb{E}}_t \left(\frac{1_{S(T)>K}}{B(T)} \right).$$

- If interest rates are deterministic, i.e. $B(t)$ is not-stochastic, then

$$v(t) = \frac{B(t)}{B(T)} \times \tilde{\mathbb{E}}_t \left(1_{S(T)>K}\right) = e^{-\int_t^T r(s)ds} \tilde{\mathbb{E}}_t \left(1_{S(T)>K}\right),$$

- We observe that:

$$\tilde{\mathbb{E}}_t \left(1_{S(T)>K}\right) = \tilde{\Pr}_t (S(T) > K),$$

  i.e. the price of the cash-or-nothing is related to the exercise probability under the risk-neutral measure.
- Remark: the term $\mathcal{N}(d_2)$ in the Black-Scholes pricing formula is related to the exercise probability under the risk-neutral measure.

Example (3. Pricing an asset-or-nothing option)

- An asset or nothing option pays $S\left(T\right)1_{S(T)>K}$.
- Given an appropriate numeraire, the option price is:

$$v\left(t\right)=num\left(t\right)\mathbb{E}_t^{num}\left(\frac{S\left(T\right)1_{S(T)>K}}{num\left(T\right)}\right).$$

- What is a numeraire that can simplify the computation?
- Let us choose as $num(t)=S(t)$, and therefore

$$v(t)=S(t)\times\mathbb{E}_t^{S}\left(\frac{S(T)1_{S(T)>K}}{S\left(T\right)}\right)=S(t)\mathbb{E}_t^{S}\left(1_{S(T)>K}\right).$$

  and we observe that:

$$\mathbb{E}_t^{S}\left(1_{S(T)>K}\right)=\Pr_t^{S}\left(S\left(T\right)>K\right),$$

  i.e. the option price is related to the exercise probability under the "spot"-measure.
- **Remark:** the term $\mathcal{N}(d_1)$ in the Black-Scholes model represents the exercise probability when the stock is taken as numeraire.

Example (4. Pricing a call option)

- A call option pays

$$(S(T)-K)^{+} = S(T)\times 1_{S(T)>K} - K \times 1_{S(T)>K}.$$

- The above payoff can be obtained as difference between:
  - a long asset-or-nothing and
  - short K cash-or-nothing options.
- Therefore the option price, assuming deterministic interest rates, is given by

$$c(t) = S(t)\times \Pr_t^S(S(T)>K) - \frac{B(t)}{B(T)}\times K \times \tilde{\Pr}_t(S(T)>K).$$

Problem (Question)

- *How to compute the probability of exercise using the two different numeraires?*
- *We examine now how to find them in the discrete time binomial model.*
- *Then we examine the Black-Scholes continuous time model.*

Example 3: Binomial model and change of numeraire

Binomial model

Stock Price Dynamics

$$\begin{array}{lll} t & t+dt & prob. \\ & Su & p \\ & \nearrow & \\ S & & \\ & \searrow & \\ & Sd & 1-p \end{array}$$

Bank Account Dynamics

$$\begin{array}{lll} t & t+dt & prob. \\ & Be^{rdt} & p \\ & \nearrow & \\ B & & \\ & \searrow & \\ & Be^{rdt} & 1-p \end{array}$$

Bank Account as Numeraire I

Relative Stock dynamics

t		$t+dt$	prob.
		$\frac{Su}{Be^{rdt}}$	p
	$\nearrow$		
$\frac{S}{B}$			
	$\searrow$		
		$\frac{Sd}{Be^{rdt}}$	$1-p$

Relative Bank Account dynamics

t		$t+dt$	prob.
		$\frac{Be^{rdt}}{Be^{rdt}}$	p
	$\nearrow$		
$\frac{B}{B}$			
	$\searrow$		
		$\frac{Be^{rdt}}{Be^{rdt}}$	$1-p$

Bank Account as Numeraire II

Fact (Probability Measure associated to the Numeraire Bank Account)

- *Martingale restriction: $\frac{S}{B}$ is a martingale, i.e. if the following restriction is satisfied*

$$\frac{S}{B} = p\frac{Su}{Be^{rdt}} + (1-p)\frac{Sd}{Be^{rdt}}$$

- *Solving wrt p we obtain the famous* **Risk-Neutral probability**

$$p = \frac{e^{rdt} - d}{u - d}$$

- *Notice that p is a probability if and only if*

$$d < e^{rdt} < u,$$

i.e. there are no arbitrage opportunities between stock and bond.

Asset as numeraire I

Relative Stock dynamics

$$
\begin{array}{cccc}
t & & t+dt & prob. \\
 & & \frac{Su}{Su} = 1 & q \\
 & \nearrow & & \\
\frac{S}{S} = 1 & & & \\
 & \searrow & & \\
 & & \frac{Sd}{Sd} = 1 & 1-q
\end{array}
$$

$$
\begin{array}{cccc}
t & & t+dt & prob. \\
 & & \frac{Be^{rdt}}{Su} & q \\
 & \nearrow & & \\
\frac{B}{S} & & & \\
 & \searrow & & \\
 & & \frac{Be^{rdt}}{Sd} & 1-q
\end{array}
$$

Asset as numeraire II

Fact (Probability Measure associated to the Numeraire Stock)

- *Martingale restriction: $\frac{B}{S}$ is a martingale if the following restriction is satisfied*

$$\frac{B}{S} = q\frac{Be^{rdt}}{Su} + (1-q)\frac{Be^{rdt}}{Sd},$$

and then we obtain the so called **SPOT probability measure**

$$q = \frac{1-de^{-rdt}}{u-d}u = e^{-rdt}u\left(\frac{e^{rdt}-d}{u-d}\right) = e^{-rdt}up.$$

- *Notice that*
  - *q is a probability iff*

$$d < e^{rdt} < u,$$

    i.e. again if there are no arbitrage opportunities between stock and bond.
  - *q is equivalent to p.*

Change of numeraire and R-N derivative

- Recall that in order to move from one probability measure p to the new one p we can use the Radon-Nykodym

$$\psi(T) = \frac{S(T)}{B(T)} \frac{B(t)}{S(t)} \tag{5}$$

- In the up state,

$$\psi^{up}(T) = \frac{S(t)\,u}{B(t)\,e^{rdt}} \frac{B(t)}{S(t)} = \frac{u}{e^{rdt}}$$

so that the probability of the up state is

$$q = p \times \psi^{up}(T) = p \times \frac{u}{e^{rdt}},$$

as we just obtained.

- In the down state

$$\psi^{down}(T) = \frac{S(t)\,d}{B(t)\,e^{rdt}} \frac{B(t)}{S(t)} = \frac{d}{e^{rdt}}$$

and then the probability of the down state is

$$1 - q = (1-p) \times \psi^{down}(T) = (1-p) \times \frac{d}{e^{rdt}}.$$

Example (1. One period binomial model)

- Let us consider the one-period binomial model with $u = 1.1, d = 1/u, r = 0.05, \delta t = 0.5, S_t = 100, K = 100$, so that

$$p = 0.60879$$

and

$$q = 0.6531.$$

- The option price is

$$\begin{aligned} c(t) &= S(t) \times \Pr_t^S (S(T) > K) - e^{-r\Delta t} \times K \times \tilde{\Pr}_t (S(T) > K) \\ &= S(t) \times q - e^{-r\Delta t} \times K \times p \\ &= 100 \times 0.6531 - e^{-0.05 \times 0.5} \times 100 \times 0.60879 \\ &= 5.9376. \end{aligned}$$

Example (2. Multi-period binomial model)

- Let us price a call option with 2 years to maturity, strike 114 and given the spot price is at 100 and the volatility is 20%, given that we use a binomial tree with 10 steps.
- The up and down factors are

$$u = e^{0.2\sqrt{\frac{2}{10}}} = 1.06528, d = e^{-0.2\sqrt{\frac{2}{10}}} = 0.93871.$$

- The annualised risk-free rate is 5%, and therefore the one period RN is

$$p = \frac{e^{0.05\times\frac{2}{10}} - d}{u-d} = \frac{1.00501 - 0.93871}{1.06528 - 0.93871} = 52.379\%,$$

- The one period SPOT measure is

$$q = p \times \frac{u}{e^{r\Delta t}} = 52.379\% \times \frac{1.06528}{1.0050} = 55.521\%.$$

Prob. Measure	Risk-Neutral	SPOT
up	52.379%	55.521%
down	47.621%	44.479%

Example (3. Stock Price distribution under the two measures)

- Let a be the smallest integer such that the option is exercised

$$Su^a d^{n-a} \geq K,$$

and we find $a = \left\lfloor \frac{\ln\left(\frac{K}{Sd^n}\right)}{ln(u/d)} \right\rfloor + 1 = \lfloor 11.03 \rfloor + 1 = 12.$

- Let us compute the probability of exercise under the spot measure

$$Pr^{spot}\left(S(n\Delta t) > K\right) = 1 - \sum_{j=0}^{a-1} \binom{n}{j} q^j (1-q)^{n-j} = 43.285\%.$$

where we have used $q = 55.019\%$, $a = 12$ and $n = 20$.

- Similarly, we have (using $p = 50.564\%$)

$$\tilde{Pr}\left(S(n\Delta t) > K\right) = 1 - \sum_{j=0}^{a-1} \binom{n}{j} p^j (1-p)^{n-j} = 32.492\%.$$

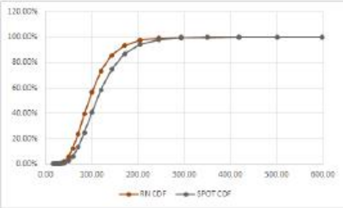
- The option price is therefore

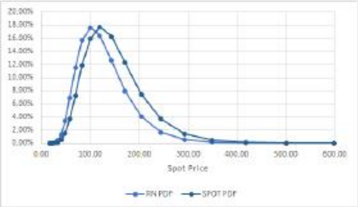
$$100 \times 43.285\% - e^{-0.05\times 2} \times 114 \times 32.492\% = 9.76901.$$

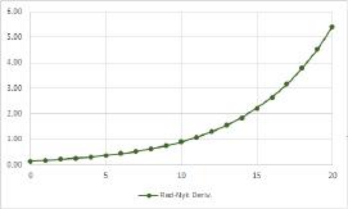
i	$S_i(T)$	$C_i(T)$	$B_i(T)$	$\frac{C_i(T)}{B_i(T)}$	Q_i^{RN}	$\frac{C_i(T)}{S_i(T)}$	Q_i^{SPOT}	$\psi_i(T)$	$Q_i^{RN}\psi_i(T)$	$1-\sum Q_i^{RN}$	$1-\sum Q_i^{S}$
20	354.28	240.28	1.11	217.41	0.00%	0.68	0.00%	3.21	0.00%	0.00%	0.00%
19	312.18	198.18	1.11	179.32	0.00%	0.63	0.01%	2.82	0.01%	0.00%	0.01%
18	275.09	161.09	1.11	145.76	0.04%	0.59	0.09%	2.49	0.09%	0.04%	0.11%
17	242.40	128.40	1.11	116.19	0.21%	0.53	0.45%	2.19	0.45%	0.25%	0.56%
16	213.60	99.60	1.11	90.12	0.80%	0.47	1.55%	1.93	1.55%	1.05%	2.11%
15	188.22	74.22	1.11	67.16	2.33%	0.39	3.96%	1.70	3.96%	3.38%	6.07%
14	165.86	51.86	1.11	46.92	5.29%	0.31	7.94%	1.50	7.94%	8.67%	14.01
13	146.15	32.15	1.11	29.09	9.62%	0.22	12.72%	1.32	12.72%	18.28%	26.73
12	128.79	14.79	1.11	13.38	14.21%	0.11	16.56%	1.17	16.56%	**32.49%**	**43.29**
11	113.48	0.00	1.11	0.00	17.22%	0.00	17.69%	1.03	17.69%	49.72%	60.97%
10	100.00	0.00	1.11	0.00	17.22%	0.00	15.59%	0.90	15.59%	66.94%	76.56
9	88.12	0.00	1.11	0.00	14.24%	0.00	11.35%	0.80	11.35%	81.18%	87.91
8	77.65	0.00	1.11	0.00	9.71%	0.00	6.82%	0.70	6.82%	90.88%	94.73
7	68.42	0.00	1.11	0.00	5.43%	0.00	3.36%	0.62	3.36%	96.31%	98.09
6	60.29	0.00	1.11	0.00	2.47%	0.00	1.35%	0.55	1.35%	98.78%	99.44
5	53.13	0.00	1.11	0.00	0.90%	0.00	0.43%	0.48	0.43%	99.68%	99.87
4	46.82	0.00	1.11	0.00	0.26%	0.00	0.11%	0.42	0.11%	99.94%	99.98
3	41.25	0.00	1.11	0.00	0.05%	0.00	0.02%	0.37	0.02%	99.99%	100.00
2	36.35	0.00	1.11	0.00	0.01%	0.00	0.00%	0.33	0.00%	100.00%	100.00
1	32.03	0.00	1.11	0.00	0.00%	0.00	0.00%	0.29	0.00%	100.00%	100.00
0	28.23	0.00	1.11	0.00	0.00%	0.00	0.00%	0.26	0.00%	100.00%	100.00

Table: Pricing an option using different measures

1. $14.65497 = 1 \times \sum_{i=12}^{20} \frac{C_i(T)}{B_i(T)} Q_i^{RN}$
2. $14.65497 = 100 \times \sum_{i=12}^{20} \frac{C_i(T)}{S(T_i)} Q_i^{SPOT}$
3. $14.65497 = 100 \times (1 - 0.4080) - \frac{114}{1.1052} \times (1 - 0.5681).$
4. Radon-Nykodyn Derivative $\psi_i(T) = \frac{S_i(T)}{B_i(T)} \frac{B(t)}{S(t)}$







$$
\begin{aligned}
v(t) &= N(t)\mathbb{E}_t^N\left[\frac{v(T)}{N(T)}\right] \\
&= M(t)\mathbb{E}_t^N\left[\frac{v(T)}{M(T)}\frac{M(T)N(t)}{N(T)M(t)}\right] \\
&= M(t)\mathbb{E}_t^N\left[\frac{v(T)}{M(T)}\psi(T)\right] \\
&= M(t)\mathbb{E}_t^M\left[\frac{v(T)}{M(T)}\right]
\end{aligned}
$$

Figure: Distribution of the stock price using different numeraires and the Radon-Nykodin derivative $\psi(T)$ that allows us to move from the RN measure to the SPOT measure.

Question

- Consider a two-periods binomial model where $S_0 = 100$, $B_0 = 1$, $u = 1.1 = 1/d$ and $r = 0.05$ (one step=1 yr).
- Consider an at-the-money call option. Construct its dynamics given that:
    - a) the asset is taken as numeraire,
    - b) the money account is the numeraire.
- What is the price of the call option in the two cases. Why?

Change of numeraire and SDE

How does the change of numeraire work in continuous time? I

- The change of numeraire is equivalent to a change of measure that can be performed by using the Radon-Nykodin derivative.
- Moreover, if we work with sde, the Girsanov theorem tells that if we are able to write the RN as an exponential martingale we can change measure via a change of drift in the sde, given technical conditions (see Appendix).
- In the following, we use this fact: when we use stochastic differential equations, if we want to change measure we have to apply a change of drift, so that the desired quantity is a martingale.
- We do this without going trough the Radon-Nykdoyn derivative, but we are aware that its knowledge and the Girsanov theorem makes this procedure meaningful.
- We do this in the Black-Scholes setting to move from the risk-neutral to the spot measure

Changing measure in the Black-Scholes model I

Recall the following facts:

1. Let

$$X \sim \mathcal{N}(m, v),$$

then its moment generating function is

$$\mathbb{E}\left(e^{\gamma X}\right) = e^{\gamma m + \frac{1}{2}\gamma^2 v}.$$

2. The sde $dS = \mu S dt + \sigma S dW$ admits as solution

$$S(T) = S(t) e^{(\mu - \sigma^2/2)(T-t) + \sigma(W(T) - W(t))}.$$

3. If $dB(t) = r(t)B(t)dt$, then

$$B(T) = B(t) e^{\int_t^T r(s)ds},$$

and if r is constant

$$B(T) = B(t) e^{r(T-t)}.$$

Changing measure in the Black-Scholes model II

Then, consider the

- Money market account dynamics

$$B(T) = B(t)e^{r(T-t)},$$

and the stock dynamics under the pricing measure

$$S(T) = S(t) \times e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{W}(T) - \tilde{W}(t))}.$$

- Here the pricing measure depends on the numeraire.
- The probability measure associated to the numeraire here is fixed once we choose the drift μ.

Changing measure in the Black-Scholes model III

- **Risk-Neutral Measure**: the money market account is the numerarie.
- Then

$$\frac{S(T)}{B(T)} = \frac{S(t)e^{(\mu-\sigma^2/2)(T-t)+\sigma(W(T)-W(t)}}{B(t)e^{r(t)(T-t)}}$$

  must be a martingale.
- This is true if and only if

$$\mu = r.$$

- **Spot Measure**: the stock is the numeraire.
- Therefore

$$\frac{B\,(T)}{S(T)} = \frac{B\,(t) \times e^{\left(r-\mu+\frac{\sigma^2}{2}\right)(T-t)-\sigma(W^S(T)-W^s(t))}}{S(t)}$$

  must be a martingale.
- This is true if only if

$$\mu = r + \sigma^2.$$

Fact (Money market account as numeraire: Risk-neutral measure)

$\frac{S(t)}{B(t)}$ *is a martingale and the stock dynamics are*

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t),$$

$$d\ln S(t) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma d\tilde{W}(t),$$

$$S(T) = S(t) \times e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{W}(T)-\tilde{W}(t))},$$

$$ln(S(T)) \sim \mathcal{N}\left(ln(S(t)) + \left(r - \frac{\sigma^2}{2}\right) \times (T-t), \sigma^2 \times (T-t)\right).$$

Fact (Stock as numeraire: Spot Measure)

$\frac{B(t)}{S(t)}$ *is a martingale and the stock dynamics are*

$$dS(t) = \left(r+\sigma^2\right) S(t)dt + \sigma S dW^S(t),$$
$$d\ln S(t) = \left(r+\frac{\sigma^2}{2}\right) dt + \sigma dW_t^S,$$
$$S(T) = S(t)e^{\left(r+\frac{\sigma^2}{2}\right)\times(T-t)+\sigma(W^S(T)-W^S(t))},$$
$$ln\left(S(T)\right) \sim \mathcal{N}\left(ln(S(t)) + \left(r+\frac{\sigma^2}{2}\right)\times(T-t), \sigma^2\times(T-t)\right).$$

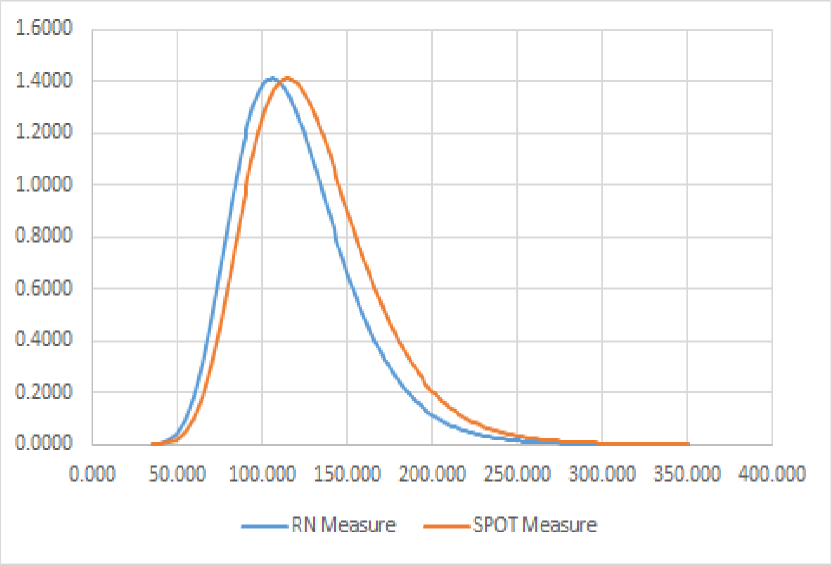


Figure: The stock price distribution in the BS model under the Risk-Neutral and the Spot measures. Parameters: $r = 5\%$, $\sigma = 20\%$, $S = 100$, $T - t = 2$.

Application: The Black-Scholes formula

- Now we price a call option with payoff $(S(T) - K)^{+}$. We assume not-risky interest rates.
- Recall that we have:

$$
\begin{aligned}
c\,(t) &= S(t)\mathbb{E}_t^S\left[\frac{S(T)1_{(S(T)>K)}}{S(T)}\right] - B(t)\mathbb{E}_t^B\left[\frac{1_{(S(T)>K)}}{B(T)}\right] \\
&= S(t)\mathbb{E}_t^S\left[1_{(S(T)>K)}\right] - \frac{B(t)}{B(T)}\mathbb{E}_t^B\left[1_{(S(T)>K)}\right] \\
&= S(t)\,\mathrm{Pr}_t^S\,(S(T)>K) - \frac{B(t)}{B(T)}K\,\mathrm{Pr}_t\,(S(T)>K)\,.
\end{aligned}
$$

- Now we have to calculate the probability of exercising the option under the two measures.

Computation of $\Pr_t\left(S(T) > K\right)$

- We know that *under the risk-neutral measure*

$$S(T) = S(t) \times e^{\left(r - \frac{\sigma^2}{2}\right)\times(T-t) + \sigma\left(\tilde{W}(T) - \tilde{W}(t)\right)},$$

- We can now compute

$$\begin{aligned}
\Pr_t\left(S(T) > K\right) &= \Pr_t\left(S(t) \times e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}(T) - \tilde{W}(t)\right)} > K\right) \\
&= \Pr_t\left(\sigma\left(\tilde{W}(T) - \tilde{W}(t)\right) > \ln\frac{K}{S(t)} - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right) \\
&= \Pr_t\left(\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{T-t}} > \frac{\ln\frac{K}{S(t)} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= 1 - \mathcal{N}\left(\frac{\ln\frac{K}{S(t)} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathcal{N}\left(-\frac{\ln\frac{K}{S(t)} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathcal{N}\left(\frac{\ln\frac{S(t)}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathcal{N}\left(d_2\right).
\end{aligned}$$

Computation of $\Pr_t^S(S(T) > K)$

- We know that *when the stock is taken as numeraire*

$$S(T) = S(t) \times e^{\left(r+\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W^S(T)-W^S(t)\right)},$$

so that we can now compute $\Pr_t^S(S(T) > K)$:

$$\begin{aligned}
&\Pr_t^S(S(T) > K) \\
&= \Pr_t^S\left(S(t)\, e^{\left(r+\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W^S(T)-W^S(t)\right)} > K\right) \\
&= \Pr_t^S\left(\sigma\left(W^S(T) - W^S(t)\right) > \ln\left(\frac{K}{S(t)}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)\right) \\
&= \Pr_t^S\left(\frac{W^S(T)-W^S(t)}{\sqrt{T-t}} > \frac{\ln\frac{K}{S(t)} - \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= 1 - \mathcal{N}\left(\frac{\ln\frac{K}{S(t)} - \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathcal{N}\left(-\frac{\ln\frac{K}{S(t)} - \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathcal{N}\left(\frac{\ln\frac{S(t)}{K} + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathcal{N}(d_1).
\end{aligned}$$

The Black-Scholes formula

- Collecting terms, we have:

$$\begin{aligned} c(t) &= S(t)\Pr_t^S(S(T)>K) - \frac{B(t)}{B(T)} K \Pr_t(S(T)>K) \\ &= S(t)\mathcal{N}(d_1) - \frac{B(t)}{B(T)} K\mathcal{N}(d_2) \\ &= S(t)\mathcal{N}(d_1) - P(t,T)\, K\mathcal{N}(d_2). \end{aligned}$$

Fact (The Black formula)

- *The Black formula is a variant of the Black-Scholes and is obtained when we consider an option on a forward contract.*
- *Let $S(t, T)$ be the forward price. The option price is given by*

$$P(t, T)\left(S(t, T)\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right),$$

where

$$d_{1,2} = \frac{ln\left(\frac{S(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}.$$

- *Notice that this formula is assuming that the dynamics of the forward price is*

$$dS(t, T) = \sigma S(t, T)d\tilde{W}(t), t \leq T,$$

i.e. it is a lognormal martingale.

- *In practice, it can be obtained by setting $r = 0$ in the expressions of d_1 and d_2 and in replacing $S(t, T)$ with $S(t)e^{r(T-t)}$ in the Black-Scholes formula.*

Problems with the m.m.a.

The problem with the m.m.a.

- In order to use the formula

$$v(t) = B(t)\tilde{\mathbb{E}}_t\left(\frac{v(T)}{B(T)}\right),$$

when interest rates are stochastic, we need to know the joint distribution of $v(T)$ and $B(T) = B(t)e^{\int_t^T r(s)ds}$.

- In general this is not easy. If the two quantities were independent, we could write:

$$v(t) = B(t)\tilde{\mathbb{E}}_t\left[\frac{1}{B(T)}\right]\tilde{\mathbb{E}}_t[v(T)] = P(t,T)\tilde{\mathbb{E}}_t[v(T)],$$

but this is not true...under the risk neutral measure.

- Let us try to reformulate the pricing problem using a new numeraire.

A zcb as numeraire: The forward measure

Forward measure I

- **Question:** There exists a new numeraire *num* (t) (and the associated probability measure), such that the computation of

$$v\left(t\right) = num(t) \times \mathbb{E}_t^{num}\left[\frac{v\left(T\right)}{num\left(T\right)}\right]$$

is simpler than calculating

$$v\left(t\right) = B(t) \times \mathbb{E}_t^{num}\left[\frac{v\left(T\right)}{B\left(t\right)e^{\int_t^T r(s)ds}}\right]$$

when interest rates are stochastic?

Forward measure II

- `Intuition`: let us set
$$num(t) = P(t, T),$$
- $P(t, T)$ is the price of a traded asset and can be taken as numeraire.
- It follows
$$v(t) = P(t, T)\,\mathbb{E}_t^{num}\left[\frac{v(T)}{P(T, T)}\right] = P(t, T)\,\mathbb{E}_t^{num}\left[\frac{v(T)}{1}\right].$$
- The associated probability measure is called T-forward measure, because the forward price $v(t)/P(t, T)$ is a martingale, indeed we have:
$$\frac{v(t)}{P(t, T)} = \mathbb{E}_t^{num}\left[\frac{v(T)}{P(T, T)}\right].$$
- Now we have to determine the distribution of the payoff under the new probability measure.
- Let us consider few examples.

Forward price of a zcb

Forward measure and forward contract on a zcb I

- The payoff of a forward contract expiring in T_1 on a zcb with maturity T_2 is

$$P(T_1, T_2) - P(t, T_1, T_2),$$

- Here the forward price $P(t, T_1, T_2)$ is fixed in t and it has to be chosen to guarantee a zero-initial cost, i.e.

$$0 = num(t) \times \mathbb{E}_t^{T_1}\left(\frac{P(T_1, T_2) - P(t, T_1, T_2)}{num(T_1)}\right),$$

  where $num(t)$ is the chosen numeraire.
- Therefore $P(t, T_1, T_2)$ must satifsy

$$\mathbb{E}_t^{T_1}\left(\frac{P(T_1, T_2)}{num(T_1)}\right) = \mathbb{E}_t^{T_1}\left(\frac{1}{num(T_1)}\right) \times P(t, T_1, T_2). \tag{6}$$

- Let us choose as numeraire the T_1-zcb:

$$num(t) = P(t, T_1)$$

Forward measure and forward contract on a zcb II

- We have

$$\mathbb{E}_t^{T_1}\left(\frac{P(T_1, T_2)}{P(T_1, T_1)}\right) = \frac{P(t, T_2)}{P(t, T_1)}, \textit{and}$$
$$\mathbb{E}_t^{T_1}\left(\frac{1}{num(T_1)}\right) = \mathbb{E}_t^{T_1}\left(\frac{1}{P(T_1, T_1)}\right) = \mathbb{E}_t^{T_1}\left(\frac{1}{1}\right) = 1,$$

where the first row exploits the fact that relative prices are martingales.

- In conclusion, substituting in (6)

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)},$$

i.e. the expression we obtained when we discussed the forward price of a bond using a no-arbitrage argument.

LIBOR in advance

Forward measure and the most important formula I

- The payoff to be priced is

$$\alpha_{1,2} \times L\left(T_1, T_2\right) \times N.$$

  and is paid in T_2, so the natural time-lag between reset and payment date applies.

- Given that $L\left(T_1, T_2\right) = F\left(T_1, T_1, T_2\right)$, and using as numeraire the $T_2 - zcb$, we have to compute

$$P\left(t, T_2\right) \times \mathbb{E}_t^{T_2}\left(\frac{F\left(T_1, T_1, T_2\right)}{P\left(T_2, T_2\right)}\right) \times \alpha_{1,2} \times N,$$

  where we are using the $T_2 - \textit{forward}$ measure.

Forward measure and the most important formula II

- Observe that
  - $P(T_2, T_2) = 1$, and
  - the simple forward rate is

$$F(t, T_1, T_2) = \frac{1}{\alpha_{T_1,T_2}} \left(\frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)} \right)$$

  and we recognize that, being a ratio of prices, the simple forward rate is a relative price: The reference asset is exactly the zcb expiring in T_2, i.e. the numeraire used for the valuation.

Forward measure and the most important formula III

- It follows:

$$
\begin{aligned}
& P(t, T_2) \times \mathbb{E}_t^{T_2}\left(\frac{F(T_1, T_1, T_2)}{P(T_2, T_2)}\right) \times \alpha_{1,2} \times N \\
& \text{given that } P(T_2, T_2) = 1 \text{ we have} \\
= \; & P(t, T_2) \times \mathbb{E}_t^{T_2}\left(F(T_1, T_1, T_2)\right) \times \alpha_{T_1,T_2} \times N \\
& \text{using the expression for the forward rate, we have} \\
= \; & P(t, T_2) \times \mathbb{E}_t^{T_2}\left(\frac{1}{\alpha_{T_1,T_2}}\left(\frac{P(T_1, T_1) - P(T_1, T_2)}{P(T_1, T_2)}\right)\right) \times \alpha_{T_1,T_2} \times N \\
& \text{we have to compute the expectation of a relative price,} \\
& \text{therefore by no-arbitrage, we have} \\
= \; & P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1,T_2} \times N
\end{aligned}
$$

Forward measure and the most important formula IV

- **FACT:** **The simple forward rate is a relative price with respect to the chosen numeraire, and then it is a martingale under the T_2 forward measure**

$$\mathbb{E}_t^{T_2}\left(F\left(T_1, T_1, T_2\right)\right) = F(t, T_1, T_2).$$

- The result is independent on the distribution of the simple forward rate: indeed, we also obtained it by a no-arbitrage argument (the most important formula).
- Whatever the dynamics of the simple forward rate, if we are using the T_2 measure, it must be guaranteed that it is martingale.
- If we are using a SDE, this is equivalent to require that the drift is zero!

Black	$dF(t, T_1, T_2) = 0dt + \sigma_{black} F(t, T_1, T_2) dW^{T_2}(t),$	$t \leq T_1$
Bachelier	$dF(t, T_1, T_2) = 0dt + \sigma_{bach} dW^{T_2}(t),$	$t \leq T_1$
Shifted Black	$dF(t, T_1, T_2) = 0dt + \sigma_{shift}(F(t, T_1, T_2) + \delta) dW^{T_2}(t),$	$t \leq T_1$

Pricing caplets

Forward measure and pricing caplets I

- The caplet payoff in T_{i+1} is

$$c(T_{i+1}) = (F(T_i, T_i, T_{i+1}) - K)^+ \alpha_{T_i, T_{i+1}}.$$

- Let us choose as numeraire the T_{i+1}-zcb

$$num(t) = P(t, T_{i+1}),$$

- We have to compute

$$c(t) = P(t, T_{i+1}) \mathbb{E}_t^{T_{i+1}} \left[\frac{(F(T_i, T_i, T_{i+1}) - K)^+}{P(T_{i+1}, T_{i+1})} \right].$$

- Therefore (notice that $P(T_{i+1}, T_{i+1}) = 1$)

$$c(t) = P(t, T_{i+1}) \mathbb{E}_t^{T_{i+1}} \left[(F(T_i, T_i, T_{i+1}) - K)^+ \right]$$

Drift restriction and Volatility coefficient

We aim to compute

$$\mathbb{E}_t^{T_{i+1}}\left[\left(F\left(T_i, T_i, T_{i+1}\right) - K\right)^+\right]$$

Notice that

- the forward rate $F(t) = F(t, T_i, T_{i+1})$ must be a martingale under the T_{i+1} probability measure;
- the martingale restriction is equivalent to have zero drift in the sde dynamics;
- no restriction is imposed on the volatility coefficient.
- Therefore, several models are admissible for $F(t)$.
- The most common are

Model	Dynamics
Bachelier	$dF(t) = \sigma \times dW^{T_{i+1}}(t), t \leq T_i$
Black	$dF(t) = \sigma \times F(t) \times dW^{T_{i+1}}(t), t \leq T_i$
Displaced Black	$dF(t) = \sigma \times (F(t) + \delta) \times dW^{T_{i+1}}(t), t \leq T_i$

Forward Measure and the Black martingale model

The Black formula for pricing caplets

1. Payoff in T_{i+1} is $c(T_{i+1}) = (F(T_i, T_i, T_{i+1}) - K)^+ \alpha_{T_i, T_{i+1}}$.
2. Use as numeraire the T_{i+1} zcb, i.e $P(t, T_{i+1})$, and compute

$$c(t) = P(t, T_{i+1}) \mathbb{E}_t^{T_{i+1}} \left((F(T_i, T_i, T_{i+1}) - K)^+ \right) \alpha_{T_i, T_{i+1}}. \tag{7}$$

3. Beyond being a martingale, we freely assume that $F(t, T_i, T_{i+1})$ is lognormal

$$F(T_i, T_i, T_{i+1}) = F(t, T_i, T_{i+1}) e^{-\frac{\sigma^2}{2}(T_i - t) + \sigma\left(W^{T_{i+1}}(T_i) - W^{T_{i+1}}(t)\right)},$$

   or equivalently $dF(t, T_i, T_{i+1}) = \sigma F(t, T_i, T_{i+1}) dW^{T_{i+1}}(t)$.
4. Computing the expectation in (7), we get the Black formula for caplets

$$P(t, T_{i+1}) (F(t, T_i, T_{i+1}) N(d_1) - K N(d_2)) \alpha_{T_i, T_{i+1}}$$

   where

$$d_{1,2} = \frac{\ln\left(\frac{F(t, T_i, T_{i+1})}{K}\right) \pm \frac{\sigma^2}{2}(T_i - t)}{\sigma\sqrt{(T_i - t)}}$$

Question

Using the Black model show that

1. The price of the LIBOR in advance is

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2}.$$

2. The price of the LIBOR in arrears is

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2} + TA(t, T_1, T_2),$$

where the timing adjustment $TA(t, T_1, T_2)$ is given by

$$\begin{aligned} TA(t, T_1, T_2) &= P(t, T_2)\, \mathbb{E}_t^{T_1}\left(F^2(T_1, T_1, T_2)\right) \alpha_{T_1, T_2}^2 && (8) \\ &= P(t, T_2) \times F^2(t, T_1, T_2) \times \alpha_{T_1, T_2}^2 \times e^{\sigma^2 (T_1 - t)}. && (9) \end{aligned}$$

3. Under the T_2 measure, the probability of exercising a caplet expiring in T_i is $\mathcal{N}(d_2)$.

Forward Measure and the Gaussian (Bachelier) martingale model

The Gaussian martingale model I

- In the last few years, we have seen the appearance of negative rates, so that the Lognormal assumption has been released.
- A possible approach is model the simple forward rate under the T_2 forward measure according to an Arithmetic Brownian Motion (Bachelier model) without drift

$$dF(t, T_1, T_2) = \sigma(t)dW^{T_2}(t), t \leq T_1.$$

- In practice, we assume that

$$F(s, T_1, T_2) = F(t, T_1, T_2) + \int_t^s \sigma(s)dW^{T_2}(s)$$

i.e.

$$F(s, T_1, T_2) \sim \mathcal{N}\left(F(t, T_1, T_2), V(t, s)\right),$$

where

$$V(t, s) = \int_t^s \sigma^2(u)du.$$

and if the diffusion coefficient is constant, $V(t, s) = \sigma^2(s - t)$.

The Gaussian martingale model II

Question

- Using this model show that
  1. The price of the LIBOR in advance is

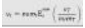
$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2}.$$

  2. The price of the LIBOR in arrears is

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1, T_2} + TA(t, T_1, T_2),$$

  where the timing adjustment $TA(t, T_1, T_2)$ is given by

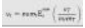
$$\begin{aligned} TA(t, T_1, T_2) &= P(t, T_2)\, \mathbb{E}_t^{T_1} \left(F^2(T_1, T_1, T_2) \right) \alpha^2_{T_1, T_2} \quad (10) \\ &= P(t, T_2) \left(F^2(t, T_1, T_2) + V(t, T_1) \right) \times \alpha^2_{T_1, T_2}. \quad (11) \end{aligned}$$



The Gaussian martingale model III

3. Under the T_2 measure, the probability of exercising a caplet expiring in T_1 is
$$\mathcal{N}(d_1), \quad \text{where } d_1 = \frac{F(t, T_1, T_2) - K}{\sqrt{V(t, T_1)}}.$$

4. The caplet price is
$$P(t, T_2) \times \left((F(t, T_1, T_2) - K)\mathcal{N}(d_1) + \sqrt{V(t, T_1)} n(d_1) \right) \times \alpha_{T_1, T_2}$$
where
$$d_1 = \frac{F(t, T_1, T_2) - K}{\sqrt{V(t, T_1)}},$$
and $n(x)$ is the standard Gaussian density.



Forward Measure and the Shifted Black Model

The Shifted Black model I

- In the last few years, we have seen the appearance of negative rates, so that the Lognormal assumption has been released.
- A possible approach is model the simple forward rate under the T_2 forward measure according to a Shifted Black Model without drift.
- We set $dX(t) = \sigma X(t) dW^{T_2}(t)$, (i.e. a GBM process), so that

$$X(s) = X(t) e^{-\frac{\sigma^2}{2}(s-t) + \sigma(W^{T_2}(s) - W^{T_2}(t))}.$$

- Then we set

$$F(t, T_1, T_2) = X(t) - \delta,$$

so that

$$F(s, T_1, T_2) = (F(t, T_1, T_2) + \delta) e^{-\frac{\sigma^2}{2}(s-t) + \sigma(W^{T_2}(s) - W^{T_2}(t))} - \delta$$

The Shifted Black model II

- In practice, we assume that

$$\ln(F(s, T_1, T_2) + \delta) = \ln(X(s)) \sim \mathcal{N}\left(M(t,s), V(t,s)\right),$$

where

$$M(t,s) = \ln(F(t, T_1, T_2) + \delta) - \frac{\sigma^2}{2}(s - t)$$

$$V(t,s) = \sigma^2(s - t).$$

- Using the Ito's lemma, we also the dynamics of the simple forward rate

$$dF(t, T_1, T_2) = \sigma(F(t, T_1, T_2) + \delta)dW^{T_2}(t), t \leq T_1.$$

The Shifted Black model III

Question

- Using this model show that
  1. The price of the LIBOR in advance is

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1,T_2}.$$

  2. The price of the LIBOR in arrears is

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1,T_2} + TA(t, T_1, T_2),$$

where the timing adjustment $TA(t, T_1, T_2)$ is given by

$$\begin{aligned} TA(t, T_1, T_2) &= P(t, T_2)\,\mathbb{E}_t^{T_1}\left(F^2(T_1, T_1, T_2)\right)\alpha^2_{T_1,T_2} && (12)\\ &= \cdots . && (13)\end{aligned}$$

  3. The caplet price is

$$P(t, T_2) \times ((F(t, T_1, T_2) + \delta)\mathcal{N}(d_1) - (K + \delta)\mathcal{N}(d_2)) \times \alpha_{T_1,T_2}$$

where

$$d_{1,2} = \frac{\ln\left(\frac{F(t,T_1,T_2)+\delta}{K+\delta}\right) \pm \frac{\sigma^2}{2}(T_i - t)}{\sigma\sqrt{(T_i - t)}}.$$

Forward Measure and pricing of bond options

Pricing options on coupon bond I

- The payoff of the option on a coupon bond is

$$\left(\sum_{i=1}^{n}\alpha_{i-1,i}\times c\times P\left(T_0,T_i\right)+P\left(T_0,T_n\right)-K\right)^+,$$

where T_0 is the option expiry ad $T_i > T_0$ are the payment dates of the bond (only cash flows occurring after the option expiry matter).

- We can rewrite it in terms of the forward bond price

$$\left(\sum_{i=1}^{n}\alpha_{i-1,i}\times c\times P\left(T_0,T_0,T_i\right)+P\left(T_0,T_0,T_n\right)-K\right)^+.$$

Pricing options on coupon bond II

- The option price is, taking the $T_0 - zcb$ as numeraire:

$$P(t, T_0) \times \mathbb{E}_t^{T_0}\left(\frac{(CB(T_0, T_0, \{T_1, ..., T_n\}) - K)^+}{P(T_0, T_0)}\right),$$

where CB is the forward price of the coupon bond

$$CB(t) = CB(t, T_0, \{T_1, ..., T_n\}) = \sum_{i=1}^{n} \alpha_{i-1,i} \times c \times P(t, T_0, T_i) + P(t, T_0, T_n).$$

- We observe that this forward price is a martingale and we assume it has a lognormal dynamics

$$dCB(t) = \sigma_{CB}(t) \times CB(t) \times dW^{T_0}(t), t \leq T_0.$$

Pricing options on coupon bond III

- Therefore, we can apply the Black formula to get:

$$P(t, T_0) \times \left(CB(t) \times \mathcal{N}(d_1) - K \times \mathcal{N}(d_2) \right)$$

  where

$$d_{1,2} = \frac{\ln\left(\frac{CB(t)}{K}\right) \pm \frac{1}{2} \times \int_t^{T_0} \sigma_{CB}^2(s)\, ds}{\sqrt{\int_t^{T_0} \sigma_{CB}^2(s)\, ds}}.$$

- Notice that there is some inconsistency in assuming that the zcb (forward) prices are lognormal and at the same time the (forward) coupon bond price to be lognormal: a sum of lognormals is not lognormal.

Forward measure and Expectation Theory

Forward Measure and Expectation Theory (ET) I

- Recall that the instantaneous forward rate is defined as

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}.$$

- ET assumes that current forward rates have some predictive power in forecasting future interest rates, i.e.

$$f(t,T) = \mathbb{E}_t(r(T)).$$

- The ET is often tested empirically using historical data and with contradictory results.
- Does it make sense to test the ET under the empirical measure?

Forward Measure and Expectation Theory (ET) II

- Let us consider:

$$P(t, T) = \tilde{\mathbb{E}}_t \left(e^{-\int_t^T r(u)du} \right)$$

and let us take the derivative with respect to T and change sign. We have

$$\begin{aligned} -\frac{\partial P(t, T)}{\partial T} &= \tilde{\mathbb{E}}_t \left(e^{-\int_t^T r(u)du} r(T) \right) \\ &= P(t, T) \mathbb{E}_t^T (r(T)) \\ &= P(t, T) \mathbb{E}_t^T (f(T, T)), \end{aligned}$$

where in last two lines we have used the change of measure and the relationship between $r(T)$ and $f(T, T)$.

- We also have

$$\begin{aligned} f(t, T) &= -\frac{\partial P(t, T)}{\partial T} \frac{1}{P(t, T)} = \mathbb{E}_t^T (f(T, T)) \\ &= \mathbb{E}_t^T (f(T, T)) \end{aligned}$$

Forward Measure and Expectation Theory (ET) III

- Therefore, no arbitrage says that the instantaneous forward rate $f(t, T)$ is a martingale under the forward measure T.
- The expectation theory says that $f(t, T) = \mathbb{E}_t(f(T, T))$, i.e. that the instantaneous forward rate is a martingale under the real world measure.
- Then the econometricians usually test the expectation theory under the empirical measure!
- No reason for which the ET should hold.

Pricing stock options with stochastic interest rates

Forward measure and pricing stock options with stochastic interest rates I

- We have to price the payoff

$$(S(T) - K)^{+}.$$

- Let us consider the option to be on the forward price rather than on the stock price, with the forward expiring at the option maturity.
  - The forward price of the stock is given by

$$S(t, T) = \frac{S(t)}{P(t, T)},$$

    and at $t = T$, we have $S(T, T) = S(T)/P(T, T) = S(T)$.
  - Therefore, we want to price the payoff

$$(S(T, T) - K)^{+}.$$

Forward measure and pricing stock options with stochastic interest rates II

- Let us consider as numeraire the zcb expiring in T, so that we have to compute

$$
\begin{aligned}
c(t) &= P(t, T)\,\mathbb{E}_t^T\left(\frac{(S(T, T) - K)^+}{P(T, T)}\right) \\
&= P(t, T)\,\mathbb{E}_t^T\left((S(T, T) - K)^+\right).
\end{aligned}
$$

- In order to compute the expectation, we make the assumption that the forward price of the stock has a GBM dynamics.
- In addition, being a relative price, it must be a martingale and therefore the forward price has zero drift

$$
dS(t, T) = \sigma_F(t, T)\,S(t, T)\,dW^T.
$$

Forward measure and pricing stock options with stochastic interest rates III

- We have to chose the volatility $\sigma_F(t, T)$. We observe that

$$\ln S(t, T) = \ln \frac{S(t)}{P(t, T)} = \ln S(t) - \ln P(t, T),$$

so that $d \ln S(t, T) = d \ln S(t) - d \ln P(t, T)$ and

$$Var(d \ln S(t, T)) = Var(d \ln S(t) - d \ln P(t, T)),$$

and

$$\sigma_F^2(t, T) = \sigma_S^2(t) + \sigma_P^2(t, T) - 2\rho\sigma_S(t)\sigma_F(t, T).$$

- Solving the sde, we have

$$S(T, T) = S(t, T)\, e^{-\frac{1}{2}\int_t^T \sigma_F^2(s,T)ds + \int_t^T \sigma_F(s,T)dW^T(s)},$$

i.e. the same expression as in the Black-Scholes model where $r = 0$ and $\sigma^2(T - t)$ has been replaced by $\int_t^T \sigma_F^2(s, T)\, ds$.

Forward measure and pricing stock options with stochastic interest rates IV

- Therefore, the option price is

$$P(t,T)\left(S(t,T)\,N(d_1) - KN(d_2)\right)$$

$$d_{1,2} = \frac{\ln\frac{S(t,T)}{K} \pm \frac{1}{2}\int_t^T \sigma_F^2(s,T)\,ds}{\sqrt{\int_t^T \sigma_F^2(s,T)\,ds}}.$$

- We can rewrite the pricing formula in terms of spot quantities

$$P(t,T)\left(\frac{S(t)}{P(t,T)}N(d_1) - KN(d_2)\right) = S(t)\,N(d_1) - P(t,T)\,KN(d_2),$$

and

$$d_{1,2} = \frac{\ln\frac{S(t)}{KP(t,T)} \pm \frac{1}{2}\int_t^T \sigma_F^2(s,T)\,ds}{\sqrt{\int_t^T \sigma_F^2(s,T)\,ds}}.$$

- Notice that we can interpret this formula as the price of an exchange option (that is discussed later on).

- Now, the main point is how to chose the zcb volatility function $\sigma_P(t, T)$.
- Clearly, it must satisfy the pull-to-maturity constraint

$$\sigma_P(T, T) = 0, \forall T \geq t$$

- We can choose $\sigma_P(T, T)$ according to the Heath-Jarrow-Morton model that relates it to the volatility function of the (instantaneous) forward rates via

$$\sigma_P(t, T) = \int_t^T \sigma_f(t, s) ds.$$

- Notice that the above constraint is satisfied.
- Possible choices for $\sigma_f(t, s)$ are
  - Constant, i.e. $\sigma_f(t, s) = \sigma$, so that

$$\sigma_P(t, T) = \sigma \times (T - t).$$

  - Exponentially decaying i.e. $\sigma_f(t, s) = \sigma e^{-\lambda(T-t)}$ so that

$$\sigma_P(t, T) = \sigma \frac{1 - e^{-\lambda(T-t)}}{\lambda}.$$

Forward rate dynamics under the same measure

Remark: How many forward measures?

- Observe that the forward rate $F(t, T_i, T_{i+1})$ is a martingale only under the measure T_{i+1}.
- So each forward rate is a martingale under a particular measure.
- In order to price more complex instruments, we need to model together rates of different maturities under the same measure (e.g. the risk neutral one, or the so called terminal measure).
- In order to reconduct the forward rate processes under the same measure, we need a change of measure and the Girsanov theorem.

Forward rates under the same measure

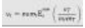
- The LIBOR market model is set up by assuming a lognormal dynamics for each forward LIBOR rate $F_i(t) = F(t, T_i, T_{i+1})$ with respect to the probability measure $\mathbb{Q}^{i+1}$.
- However, to price exotic derivatives, we need to model all LIBOR rates under the same measure. In general, it is convenient to use the terminal $\mathbb{Q}^{N+1}$ measure
- It can be shown that $t \leq s \leq T_i$, $F_i(s)$ has sde under the terminal measure $\mathbb{Q}^{N+1}$

$$
\begin{aligned}
dF_i(s) &= -\sum_{k=1+i}^{N} \frac{\sigma_i(s) F_i(s) \sigma_k(s) F_k(s) \alpha_{k,k+1}}{1 + F(s, T_k, T_{k+1}) \alpha_{k,k+1}} dt \\
&\quad + \sigma_i(s) F_i(s) dW^{N+1}(s).
\end{aligned}
$$

- Therefore, apart from $F_N(t)$, all forward Libor rates are no longer martingales under the terminal measure, but have a drift that depends on the forward Libor rates with longer maturities.

Remarks

1. The set of sde's (14) for $i = 1, \cdots, N$ represents the **Forward LIBOR Market Model**.
2. The implementation can be performed via Monte Carlo simulation, because we need to simulate all forward rates at the same time.
3. The model calibration is straightforward: the volatilities σ_i are obtained by bootstrapping the term structure of volatilities of cap prices. Unfortunately, we cannot fit the full smile and for this we need stochastic volatility models.
4. Brigo and Mercurio & Veronesi (section 22.6) discuss the extension to the multifactor version that allows for a not perfect correlation among the Brownian motions driving the F_j and F_i forward rates.
5. The Bachelier version is obtained replacing everywhere $\sigma_i F_i$ by σ_i.
6. Brigo and Mercurio, pagg. 198-203, discuss also the problems of modelling $F_i(t)$ under the risk neutral measure, that is required for some product like Eurodollar futures.
7. In general, these set of sde's do not admit a closed form solution, so we have to use numerical methods such as Monte Carlo method to solve it. See Pelsser chapter 8.



Pricing swaptions: The Annuity as numeraire and the Swap measure

Swap measure and swaption pricing I

- A swaption pays at maturity

$$c\left(T\right) = \left(S\left(T, T_0, T_n\right) - K\right)^{+} \sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} P\left(T, T_i\right).$$

- We need a numeraire $num\left(t\right)$ that makes easy the computation of

$$c\left(t\right) = num\left(t\right) \mathbb{E}_t^{num}\left[\frac{\left(S\left(T, T_0, T_n\right) - K\right)^{+} \sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} P\left(T, T_i\right)}{num\left(T\right)}\right].$$

- It can be convenient to chose

$$num\left(t\right) = \sum_{i=1}^{n} \alpha_{t, T_i} P\left(t, T_i\right).$$

- The associated probability measure is called `swap measure` (Jamshidian, 1996) and is denoted by $\mathbb{S}$.

Swap measure and swaption pricing II

- Therefore

$$c(t) = \left(\sum_{i=1}^{n} \alpha_{T_{i-1},T_i} P(t, T_i)\right) \mathbb{E}_t^{\mathbb{S}}\left[(S(T, T_0, T_n) - K)^+\right].$$

Problem

What is the distribution of $S(T, T_0, T_n)$ under the swap−measure?

$$dS(t) = 0\,dt + \left\{ \begin{array}{l} \sigma_{BAC} \\ \sigma_{BLS}(t) \\ \sigma_{SB}(S(t)+\delta) \end{array} \right.$$

Bachelier:

$$dS(t) = \sigma\, dW_t^S$$

$$C(t) = A(t)\, \mathbb{E}_t^S\left[(S(T)-K)^+\right]$$

Swap measure and swaption pricing III

- The forward swap rate is given by

$$S\left(T, T_0, T_n\right) = \frac{P\left(T, T_0\right) - P\left(T, T_n\right)}{\sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} P\left(T, T_i\right)},$$

  and so, being $S\left(T, T_0, T_n\right)$ is a relative price with respect to the annuity, it has to be a martingale.

- We need to fix its dynamics. The most common are

Model	Dynamics	
Bachelier	$dS(t) = \sigma \times dW^A(t)$,	$t \leq T$
Black	$dS(t) = \sigma \times S(t) \times dW^A(t)$,	$t \leq T$
Displaced Black	$dS(t) = \sigma \times (S(t) + \delta) \times dW^A(t)$,	$t \leq T$

Swap measure and swaption pricing IV

- For example, if we assume that S is lognormal

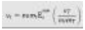
$$dS\left(t, T_0, T_n\right) = \sigma_S S\left(t, T_0, T_n\right) dW^{\mathbb{S}}, t \leq T_0.$$

- Therefore, the Black formula for swaptions is valid

$$c\left(t\right) = \left(\sum_{i=1}^{n} \alpha_{T_{i-1}, T_i} P\left(t, T_i\right)\right) \times \left(S\left(t, T_0, T_n\right) \mathcal{N}\left(d_1\right) - K\mathcal{N}\left(d_2\right)\right),$$

with

$$d_{1,2} = \frac{ln\left(\frac{S(t, T_0, T_n)}{K}\right) \pm \frac{1}{2}\sigma_S^2(T-t)}{\sqrt{\sigma_S^2(T-t)}}.$$



Question

Black Model can not be hold at the same time for both forward measure and swap measure.

What is the swaption price if the forward swap rate is assumed to be a Gaussian martingale (Bachelier model)

$$dS\left(T, T_0, T_n\right) = \sigma_S dW^{\mathbb{S}}?$$

Swap vs Forward measure

- `Problem`: Each forward rate $F(t, T_1, T_2)$ is a lognormal r.v. under its own measure "T_2". Are they still lognormal under the swap measure?
- `Problem`: The swap rate is a lognormal r.v. under the swap measure. Swap rate is an average of forward rates. What is its distribution under the forward measure?
- Brigo and Mercurio, pagg. 227-229, discuss the dynamics of forward LIBOR rates under the swap measure and viceversa and the problems related to assume a lognormal distribution. The incompatibility seems to be mostly theoretical.
- Indeed, if forward rates are higly correlated, the swap rate will be still approximately lognormal also under a change of measure.
- Using this fact, Brace, Gaterek and Musiela give an approximate Black formula for swaptions, see Brigo and Mercurio.

Spot measure

Spot measure and exchange option I

- The payoff is
$$\boxed{\left(S_1(T) - S_2(T)\right)^+}.$$
- If we take the stock 1 as numeraire and define the martingale
$$Z(t) = \frac{S_2(t)}{S_1(t)},$$
the price of the exchange option is
$$S_1(t) \times \mathbb{E}_t^{S_1}\left(\frac{\left(S_1(T) - S_2(T)\right)^+}{S_1(T)}\right) = S_1(t) \times \mathbb{E}_t^{S_1}\left(\left(1 - Z(T)\right)^+\right).$$

Spot measure and exchange option II

- This is a put option on Z and if we assume it is lognormal, i.e.

$$dZ\left(t\right)=\sigma_{Z}Z\left(t\right)dW^{S_{1}}\left(t\right),$$

we get the Black formula

$$S_{1}\left(t\right)\times\left(1\times N\left(-d_{2}\right)-Z\left(t\right)\times N\left(-d_{1}\right)\right),$$

where

$$d_{1,2}=\frac{\ln\left(Z\left(t\right)\right)\pm\frac{1}{2}\times\sigma_{Z}^{2}\times\left(T-t\right)}{\sqrt{\sigma_{Z}^{2}\times\left(T-t\right)}}.$$

- Using the definition of $Z\left(t\right)$ and replacing in the above formula, we get the so-called Margrabe formula for an exchange option

$$S_{1}\left(t\right)\times N\left(-d_{2}\right)-S_{2}\left(t\right)\times N\left(-d_{1}\right),$$

where

$$d_{1,2}=\frac{\ln\left(\frac{S_{1}(t)}{S_{2}(t)}\right)\pm\frac{1}{2}\times\sigma_{Z}^{2}\times\left(T-t\right)}{\sqrt{\sigma_{Z}^{2}\times\left(T-t\right)}}.$$

Spot measure and exchange option III

- How do we choose the volatility σ_Z?
- Given that $Z(t) = S_2(t)/S_1(t)$, we have

$$\ln Z(t) = \ln S_2(t) - \ln S_1(t),$$

and then

$$d\ln Z(t) = d\ln S_2(t) - d\ln S_1(t),$$

and then

$$Var(d\ln Z(t)) = Var(d\ln S_2(t) - d\ln S_1(t)),$$

i.e.

$$\sigma_z^2 dt = \sigma_2^2 dt + \sigma_1^2 dt - 2\rho\sigma_1\sigma_2 dt,$$

where $\sigma_i^2 = Var(d\ln S_i(t))$ and

$$\rho = Corr(d\ln S_1(t), d\ln S_2(t)).$$

- Therefore

$$\sigma_z^2 = \sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2.$$

Spot measure and exchange option IV

- The option price is, ceteris paribus, higher as the correlation between the two stocks is lower.
- Indeed, larger the volatility of the spread greater the option value.

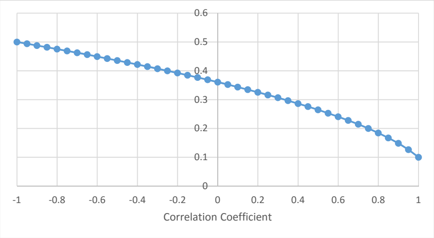


Figure: Volatility σ_z of the spread varying the correlation coefficient $(\sigma_1 = 0.3, \sigma_2 = 0.2)$

Spot measure and exchange option V

- Instead, the effect of an increase of the volatility of one stock depends on the sign and on the value of ρ.
  1. if ρ is negative, larger σ_i greater the volatility of the spread
  2. if ρ is positive and large, larger σ_i then the spread volatility at first decreases and then increases.
  3. if ρ is positive and small, larger σ_i larger the spread volatility.

Spot measure and exchange option VI

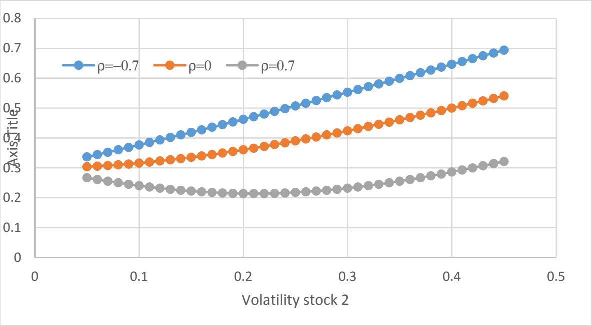


Figure: Volatility σ_z of the spread varying the volatility of stock 2 ($\sigma_1 = 0.3$).

Other Applications

- Using the change of numeraire technique, we can see several applications, such as
  - Forward LIBOR market model where we find the process of all forward rates under the same measure.
  - Forward Swap market model: how to write the process of the forward swap rate under the risk neutral measure or under the forward measure?
  - Convexity adjustment for Constant Maturity Swaps.

Appendix

Procedure

- Given that we know the dynamics of $F_{t,T_i,T_{i+1}}$ under the measure $\mathbb{T}_{i+1}$, how to find its process under a different measure?
- Two steps:
  1. how to go from a measure $\mathbb{P}$ to the new measure $\mathbb{Q}$?
     - Find the Radon-Nykodim derivative $d\mathbb{Q}/d\mathbb{P}$.
  2. how to find the distribution of F under the new measure?
     - If the uncertainty is described by a Brownian process, we can use Girsanov Theorem.
     - The change of measure results in a change of drift.

Girsanov's Theorem I

- The Change of numeraire allows us to calculate the expected value under a given measure computing an adjusted expected value under a different measure.
- In order to make the result useful, we should be able to compute the new expected value.
- The Girsanov theorem provides a tool to determine the effect of a change of measure on a stochastic process.

Girsanov's Theorem II

Fact (Girsanov Theorem)

For any stochastic process $k(t)$ *such that (Novikov condition):*

$$\mathbb{E}_t\left[\exp\left(-\frac{1}{2}\int_t^T k^2(s)\,ds\right)\right] < \infty,$$

with probability one, consider the Radon-Nykodyn derivative $d\mathbb{Q}^M/d\mathbb{Q}^N =: \psi(T)$

$$\psi(T) = \exp\left\{-\frac{1}{2}\int_t^T k^2(s)\,ds + \int_t^T k(s)\,dW^N(s)\right\}, \tag{14}$$

where W^N *is a Brownian motion under the measure* $\mathbb{Q}^N$*. Define*

$$W^M(t) = W^N(t) - \int_0^t k(s)\,ds$$

$W^M(t)$ *is also a Brownian motion under* $\mathbb{Q}^M$*.*

Girsanov's Theorem III

- We can also write in differential form

$$dW^M(t) = dW^N(t) - k(t)\,dt,$$

or

$$dW^N(t) = dW^M(t) + k(t)\,dt,$$

- If we have the sde

$$dX = m(X,t)dt + s(X,t)dW^N(t),$$

we can write the dynamics of X under the new probability measure by replacing $dW^N(t)$ by $dW^M(t)$, so that

$$dX = \mu(X,t)dt + \sigma(X,t)dW^N(t) = \mu(X,t)dt + \sigma(X,t)\left(dW^M(t) + k(t)dt\right),$$

so that the new process is

$$dX = \left(\mu(X,t) + \sigma(X,t)k(t)\right)dt + \sigma(X,t)dW^M(t),$$

i.e. we have a change of drift.

The receipe

- Start with the process under the measure $\mathbb{Q}^N$ and the corresponding numeraire.
- Choose a new numeraire M and compute the Radon-Nykodym derivative as a ratio of numeraires.
- Write the Radon-Nykodyn derivative in the form (14) and identify the process $k(t)$.
- Using $k(t)$ define a new process dW^M, i.e. a Brownian motion under the new measure.
- Replace dW^N in the risk-neutral process with $dW^M + kdt$ in the new measure.
- The change of measure results to be a change of drift, while the volatility turns out to stay the same under both probability measures

$$\mu dt + \sigma dW^M \Rightarrow (\mu + k\sigma)\, dt + \sigma dW^N.$$

Meaning of $k(t)$

- Process under the measure $\mathbb{Q}^M$:

$$\frac{dX}{X} = \mu^M dt + \sigma dW^M,$$

- Process under the measure $\mathbb{Q}^N$:

$$\frac{dX}{X} = \mu^N dt + \sigma dW^N,$$

  where

$$\mu^N = \mu^M + k\sigma$$

  and therefore

$$k = \frac{\mu^N - \mu^M}{\sigma},$$

  i.e. the risk-adjusted expected excess return.

- For example if $\mathbb{Q}^M$ is the risk-neutral measure, $\mu^M = r_f$, and $\mathbb{Q}^N$ is the empirical measure, k is the risk-premium.

Application 1: Black-Scholes Model

- Under the risk-neutral measure, we have:

$$
\begin{aligned}
dB &= rBdt, \\
dS &= rSdt + \sigma S d\tilde{W}_t.
\end{aligned}
$$

- But what is the process and the distribution of S under the spot measure?
- We now follow the receipt:
  1. write the Radon-Nykodyn derivative as a numeraire ratio;
  2. use Girsanov theorem to identify the drift adjustment $k(t)$:
  3. define a new BM under the new measure.

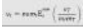
Use the Receipt I

- The Radon-Nikodym derivative is:

$$\begin{aligned}\frac{d\mathbb{Q}^S}{d\mathbb{Q}^B} &= \frac{S(T)}{B(T)}\frac{B(t)}{S(t)} \\ &= \frac{S(t)\, e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(\tilde{W}(T)-\tilde{W}_t\right)}}{e^{rT}}\frac{e^{rt}}{S(t)} \\ &= e^{-\frac{\sigma^2}{2}(T-t)+\sigma\left(\tilde{W}(T)-\tilde{W}_t\right)}.\end{aligned}$$

- To apply the Girsanov theorem, let us consider the R-N derivative and try to identify the function $k(t)$:

$$\begin{aligned}\frac{d\mathbb{Q}^S}{d\mathbb{Q}^B} &= e^{-\frac{\sigma^2}{2}(T-t)+\sigma\left(\tilde{W}(T)-\tilde{W}_t\right)} \\ &= \exp\left\{-\frac{1}{2}\int_t^T k^2(s)\,ds + \int_t^T k(s)\,d\tilde{W}(s)\right\}.\end{aligned}$$



Use the Receipt II

- So we should chose $k(t)$ such that:

$$e^{-\frac{1}{2}\int_t^T k^2(s)ds+\int_t^T k(s)d\tilde{W}(s)} = e^{-\frac{\sigma^2}{2}(T-t)+\sigma\left(\tilde{W}(T)-\tilde{W}_t\right)}$$
$$\Rightarrow \boxed{\boxed{k(t) = \sigma, \quad \forall t.}}$$

- Therefore, by the Girsanov Theorem we can define a new process dW_t^S :

$$dW_t^S = d\tilde{W}_t - \sigma dt,$$

that is a BM under the measure $\mathbb{Q}^S$.

The dynamics of S under $\mathbb{Q}^S$

- The function $k(t)$ allows us to change measure.
- From a practical aspect we can write the driving Brownian process as

$$d\tilde{W} = dW^S + \sigma dt.$$

- The asset price process under the new measure is given by:

$$\begin{aligned} dS &= rSdt + \sigma Sd\tilde{W}_t \\ &= rSdt + \sigma S\left(dW^S + \sigma dt\right) \\ &= \underbrace{\left(r + \sigma^2\right)}_{\text{change of drift}} Sdt + \sigma SdW^S. \end{aligned}$$

- The positive aspect is that S under the new measure has again a lognormal distribution:

$$\begin{aligned} S(T) &= S_t e^{\left(r+\sigma^2-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W(T)^S-W_t^S\right)} \\ &= S_t e^{\left(r+\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W(T)^S-W_t^S\right)}. \end{aligned}$$

Pricing LIBOR in arrears

Forward measure and pricing LIBOR in arrears I

- In a LIBOR in arrears contract at time T_1 we get the amount

$$L(T_1, T_2) \times \alpha_{T_1,T_2},$$

  whilst in a standard contract the above payoff is paid in T_2.
- We can rewrite the payoff using the forward rate

$$F(T_1, T_1, T_2) \times \alpha_{T_1,T_2}.$$

- Now let us move the payoff to time T_2

$$\begin{aligned} & F(T_1, T_1, T_2) \times \alpha_{T_1,T_2} \times (1 + F(T_1, T_1, T_2)\,\alpha_{T_1,T_2}) \\ & = F(T_1, T_1, T_2) \times \alpha_{T_1,T_2} + F^2(T_1, T_1, T_2)\,\alpha^2_{T_1,T_2}. \end{aligned}$$

- The market value of the first term is just (use the fundamental recipe)

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1,T_2}.$$

Forward measure and pricing LIBOR in arrears II

- In order to price the second component let us chose the $T_2 - zcb$ as numeraire. The market value of the second component becomes

$$P(t, T_2) \times \mathbb{E}_t^{T_2}\left(\frac{F^2(T_1, T_1, T_2)}{P(T_2, T_2)}\right) \times \alpha^2_{T_1,T_2}$$
$$= P(t, T_2) \times \mathbb{E}_t^{T_2}\left(F^2(T_1, T_1, T_2)\right) \times \alpha^2_{T_1,T_2}.$$

- We know that under the T_2 measure the forward rate $F(t, T_1, T_2)$ is a martingale.
- The difference with respect to a standard contract is given by the so called *timing adjustment*:

$$P(t, T_2) \times \mathbb{E}_t^{T_2}\left(F^2(T_1, T_1, T_2)\right) \times \alpha^2_{T_1,T_2}.$$

Forward measure and pricing LIBOR in arrears III

- In addition, if we assume that the forward rate is lognormal, we can write

$$dF\left(t,T_1,T_2\right)=\sigma F\left(t,T_1,T_2\right)dW^{T_2}\left(t\right),$$

or

$$F\left(T_1,T_1,T_2\right)=F\left(t,T_1,T_2\right)e^{-\frac{1}{2}\sigma^2\left(T_1-t\right)+\sigma\left(W^{T_2}\left(T_1\right)-W^{T_2}\left(t\right)\right)}.$$

- Therefore

$$F^2\left(T_1,T_1,T_2\right)=F^2\left(t,T_1,T_2\right)e^{-\sigma^2\left(T_1-t\right)+2\sigma\left(W^{T_2}\left(T_1\right)-W^{T_2}\left(t\right)\right)},$$

and

$$\begin{aligned}
&\mathbb{E}_t^{T_2}\left(F^2\left(T_1,T_1,T_2\right)\right)\\
&=\mathbb{E}_t^{T_2}\left(F^2\left(t,T_1,T_2\right)\times e^{-\sigma^2\left(T_1-t\right)+2\sigma\left(W^{T_2}\left(T_1\right)-W^{T_2}\left(t\right)\right)}\right)\\
&=F^2\left(t,T_1,T_2\right)\times e^{-\sigma^2\left(T_1-t\right)}\times\mathbb{E}_t^{T_2}\left(e^{2\sigma\left(W^{T_2}\left(T_1\right)-W^{T_2}\left(t\right)\right)}\right)\\
&=F^2\left(t,T_1,T_2\right)\times e^{-\sigma^2\left(T_1-t\right)}\times e^{\frac{4}{2}\sigma^2\left(T_1-t\right)}\\
&=F^2\left(t,T_1,T_2\right)\times e^{\sigma^2\left(T_1-t\right)}.
\end{aligned}$$

Forward measure and pricing LIBOR in arrears IV

(where in the computation of the expectation in the third line, we have used the moment generating function of a Gaussian random variable).

- In conclusion, we have that the $t-$value of the T_1-payment $L(T_1, T_2)\,\alpha_{T_1,T_2}$ is

$$P(t, T_2) \times F(t, T_1, T_2) \times \alpha_{T_1,T_2} \times \left(1 + F(t, T_1, T_2) \times e^{\sigma^2(T_1-t)} \times \alpha_{T_1,T_2}\right).$$

- The timing adjustment is given by

$$P(t, T_2) \times \mathbb{E}_t^{T_2}\left(F^2(T_1, T_1, T_2)\right) \times \alpha^2_{T_1,T_2} \times e^{\sigma^2(T_1-t)}.$$

- Notice that if the volatility is not constant, but time varying in a deterministic way, we have to replace

$$\sigma^2 \times (T_1 - t),$$

with

$$\int_t^{T_1} \sigma^2(u)du,$$

as a consequence of the isometry property of the Brownian motion

Pricing FRA in arrears

Forward measure and pricing a FRA in arrears I

- In a FRA in arrears, we receive at time T_1 the amount

$$\left(L\left(T_1, T_2\right) - K\right) \times \alpha_{T_1,T_2}.$$

- The present value of the floating amount has been obtained before. The present value of the fixed amount is

$$P\left(t, T_2\right) \times K \times \alpha_{T_1,T_2}.$$

- Or, moving it forward to time T_2 we get

$$K \times \alpha_{T_1,T_2} \times \left(1 + F\left(T_1, T_1, T_2\right) \times \alpha_{T_1,T_2}\right),$$

- Its fair value considering as numeraire the T_2 zcb is

$$P\left(t, T_2\right) \times K \times \alpha_{T_1,T_2} \times \left(1 + F\left(t, T_1, T_2\right) \times \alpha_{T_1,T_2}\right).$$

Forward measure and pricing a FRA in arrears II

- The present value of the FRA in arrears is therefore

$$P(t, T_2) \times \left(F(t, T_1, T_2) \times \left(1 + F(t, T_1, T_2) \times e^{\sigma^2 (T_1 - t)} \times \alpha_{T_1, T_2} \right) \right) \times \alpha_{T_1, T_2}$$
$$- P(t, T_2) \times K \times (1 + F(t, T_1, T_2) \times \alpha_{T_1, T_2}) \times \alpha_{T_1, T_2}.$$

- The FRA has zero value if

$$F(t, T_1, T_2) \times \left(1 + F(t, T_1, T_2) \times e^{\sigma^2 (T_1 - t)} \times \alpha_{T_1, T_2} \right)$$
$$= K \times (1 + F(t, T_1, T_2) \times \alpha_{T_1, T_2}),$$

i.e. if

$$K = F(t, T_1, T_2) \times \frac{1 + F(t, T_1, T_2) \times e^{\sigma^2 (T_1 - t)} \times \alpha_{T_1, T_2}}{1 + F(t, T_1, T_2) \times \alpha_{T_1, T_2}}. \qquad (15)$$

- Notice that, given that the term $e^{\sigma^2 (T_1 - t)} > 0$, we always have

$$K^{arrears} > F(t, T_1, T_2) = K^{advance}.$$

Pricing options on a zero-coupon bond

Forward measure and pricing options on a zcb I

- The payoff of the T_0-option on a T_1-zcb is

$$(P(T_0, T_1) - K)^+$$

(here T_0 is the option expiry; exercising the option we get a zcb with expiry in T_1, $T_1 > T_0$.)

- We can obtain different pricing formula depending on our modelling assumption
  1. the **simple forward rate is lognormal** and then we exploit the equivalence between a call option on a zcb and an appropriate number of floorlets (a variant of this is to assume that the forward rate is Gaussian), OR
  2. the **zcb forward price is lognormal**.

Forward measure and pricing options on a zcb II

Method 1: the forward rate is a lognormal martingale

- If we use

$$K = \frac{1}{1 + L_k \times \alpha_{T_0,T_1}},$$

and

$$P(T_0, T_1) = \frac{1}{1 + L(T_0, T_1) \times \alpha_{T_0,T_1}},$$

we have the following payoff in T_0

$$\begin{aligned}
& (P(T_0, T_1) - K)^+ \\
= & \left(\frac{1}{1 + L(T_0, T_1) \times \alpha_{T_0,T_1}} - \frac{1}{1 + L_k \times \alpha_{T_0,T_1}} \right)^+ \\
= & \frac{1}{1 + L(T_0, T_1) \times \alpha_{T_0,T_1}} \times \frac{\alpha_{T_0,T_1}}{1 + L_k \times \alpha_{T_0,T_1}} \times (L_k - L(T_0, T_1))^+ \\
= & P(T_0, T_1) \times \frac{\alpha_{T_0,T_1}}{1 + L_k \times \alpha_{T_0,T_1}} \times (L_k - L(T_0, T_1))^+ .
\end{aligned}$$

Forward measure and pricing options on a zcb III

- This is equivalent to the T_1 payoff

$$\frac{\alpha_{T_0,T_1}}{1+L_k\times\alpha_{T_0,T_1}}\times\left(L_k-L\left(T_0,T_1\right)\right)^+ .$$

i.e. to $\frac{\alpha_{T_0,T_1}}{1+L_k\times\alpha_{T_0,T_1}}$ floorlets that can be priced using the Black formula

$$P\left(t,T_1\right)\times\frac{\alpha_{T_0,T_1}}{1+L_k\times\alpha_{T_0,T_1}}\times\left(L_k\times N\left(-d_2\right)-L\left(T_0,T_1\right)\times N\left(-d_1\right)\right)$$

where

$$d_{1,2}=\frac{\ln\left(\frac{L(T_0,T_1)}{L_k}\right)\pm\frac{1}{2}\times\int_t^{T_0}\sigma^2(s)}{\sqrt{\int_t^{T_0}\sigma^2(s)}}.$$

- We can also have the variant where the forward rate is a Gaussian martingale.

Forward measure and pricing options on a zcb IV

Method 2: the forward price is a lognormal martingale

- We rewrite the payoff of the option on the zcb using the forward price on the zcb

$$\left(P\left(T_0, T_0, T_1\right) - K\right)^+,$$

where

$$P\left(t, T_0, T_1\right) = \frac{P\left(t, T_1\right)}{P\left(t, T_0\right)}.$$

- If we consider as numeraire the T_0-zcb the option price is

$$\begin{aligned} & P\left(t, T_0\right) \times \mathbb{E}_t^{T_0}\left(\frac{\left(P\left(T_0, T_0, T_1\right) - K\right)^+}{P\left(T_0, T_0\right)}\right) \\ = \quad & P\left(t, T_0\right) \times \mathbb{E}_t^{T_0}\left(\left(P\left(T_0, T_0, T_1\right) - K\right)^+\right). \end{aligned}$$

- The forward price is a martingale being a relative price with respect to the numeraire.

Forward measure and pricing options on a zcb V

- In addition, if we assume $P(t, T_0, T_1)$ to be lognormal with dynamics

$$dP(t, T_0, T_1) = \sigma_P(t, T_0, T_1) \times P(t, T_0, T_1) \times dW^{T_0}(t),$$

we can apply the Black formula to get the option price

$$P(t, T_0) \times (P(t, T_0, T_1) \times \mathcal{N}(d_1) - K \times \mathcal{N}(d_2)),$$

where

$$d_{1,2} = \frac{\ln\left(\frac{P(t,T_0,T_1)}{K}\right) \pm \frac{1}{2} \times \int_t^{T_0} \sigma_P^2(u, T_0, T_1)\, du}{\sqrt{\int_t^{T_0} \sigma_P^2(u, T_0, T_1)\, du}}.$$

- If the volatility of the forward price is constant, then

$$\sqrt{\int_t^{T_0} \sigma_P^2(u, T_0, T_1)\, du} = \sigma_P^2 \times (T_0 - t).$$

Remarks on pricing options on zcb

- The two formulas differ because:
  1. in the first case we assume the simple forward rate to be a martingale lognormal under the T_1-measure and σ refers to the percentage volatility of the forward rate (and can be assumed to be constant);
  2. in the second case we assume the forward price to be a martingale lognormal under the T_0-measure and σ_P refers to the percentage volatility of the forward price (and cannot be assumed to be constant).
- Given that

$$P(t, T_0, T_1) = \frac{1}{1 + F(t, T_0, T_1) \times \alpha_{T_0, T_1}},$$

  the lognormality of F does not imply the lognormality of P.
- Moreover, they are lognormal r.v's under different measures:
- In general, it is more convenient to model directly interest rates because
  - we do not need to guarantee the pull to maturity constraint and
  - if we model them as lognormal r.v.'s we can guarantee that they remain always positive.
  - if we prefer to have negative rates, we can shift to the Gaussian martingale forward rate model.

Modelling zcb prices: the HJM model I

- An important question is if

$$dP\left(t,T_0,T_1\right)=\sigma_P\left(t,T_0,T_1\right)\times P\left(t,T_0,T_1\right)\times dW^{T_0}\left(t\right) \tag{16}$$

is a valid model.

- The answer is positive and the model is the Gaussian Heath-Jarrow-Morton model.
- In this model, instantantaneous forward rates are modelled according to

$$\begin{aligned} df\left(t,T\right) &= Drift^{num}(t,T)\times dt+\sigma_f(t,T)dW^{num}(t), & (17)\\ f(0,T) &= f^{mkt}(0,t), & (18)\end{aligned}$$

for $0<t<T$.

- This is an infinite dimensional model because we have to model forward rates for all possible values of T.
- A particular HJM model is only specified once $f(0,T)$ and $\sigma_f(t,T)$ have been specified.
- Then the drift is chosen according so that, using different numeraires, relative prices are martingale.

Modelling zcb prices: the HJM model II

- In particular, we have

Measure	Numeraire	Drift($df(t,T)$)	Drift($dP(t,T)$)
Risk-Neutral	MMA	$\sigma(t,T)\int_t^T \sigma(t,u)du$	$r(t)P(t,T)$
Forward S	S-zcb	$\sigma(t,T)\int_t^S \sigma(t,u)du$	

- Then the dynamics of forward prices in (16) holds and

$$\sigma_P(t, T_0, T_1) = \sigma_P(t, T_0) - \sigma_P(t, T_1),$$

where $\sigma_P(u,T)$ refers to the percentage volatility of the zcb price and is related to the volatility of instantaneous forward rates trough

$$\sigma_P(t,T) = \int_0^T \sigma_f(t,u)du.$$

- The inverse relationship also holds

$$\sigma_f(t,T) = \frac{\partial \sigma_P(t,T)}{\partial T}.$$

Modelling zcb prices: the HJM model III

- Popular choices of the volatility function in the Gaussian HJM model are

Table: Volatility specification in two popular Gaussian one factor Heath-Jarrow-Morton model

Model	$\sigma_f(t,T)$	$\sigma_P(t,T)$	$\sigma_P(t,T_0,T_1)$
Ho and Lee	σ	$\sigma \times (T-t)$	$\sigma \times (T_1 - T_0)$
Hull and White	$\sigma e^{-\lambda(T-t)}$	$\sigma \frac{1-e^{-\lambda(T-t)}}{\lambda}$	$\sigma \frac{1-e^{-\lambda(T_1-t)}}{\lambda} - \sigma \frac{1-e^{-\lambda(T_0-t)}}{\lambda}$

and in the Table above we have

$$\begin{aligned} SDev(df(t,T)) &= \sigma_f(t,T), \\ SDev(dP(t,T)) &= \sigma_P(t,T) = \int_t^T \sigma_f(t,u)du, \\ SDev(dP(t,T_0,T_1)) &= \sigma_P(t,T_1) - \sigma_P(t,T_0) = \int_{T_0}^{T_1} \sigma_f(t,u)du. \end{aligned}$$

Pricing caplets in Gaussian HJM models

Caplet pricing in Gaussian HJM models

- The caplet payoff at time T_{i+1} is given by:

$$\alpha_{T_i,T_{i+1}}\left(F\left(T_i,\,T_i,\,T_{i+1}\right)-L_x\right)^+$$

- Consider as numeraire the zcb expiring in T_{i+1}. The caplet price is

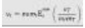
$$\begin{aligned}
&\alpha_{T_i,T_{i+1}}P\left(t,\,T_{i+1}\right)\mathbb{E}_t^{T_{i+1}}\left[\left(F\left(T_i,\,T_i,\,T_{i+1}\right)-L_x\right)^+\right]\\
=\;& P\left(t,\,T_{i+1}\right)\mathbb{E}_t^{T_{i+1}}\left[\left(\left(\frac{P\left(T_i,\,T_i\right)}{P\left(T_i,\,T_{i+1}\right)}-1\right)-\alpha_{T_i,T_{i+1}}L_x\right)^+\right]\\
=\;& P\left(t,\,T_{i+1}\right)\mathbb{E}_t^{T_{i+1}}\left[\left(\frac{P\left(T_i,\,T_i\right)}{P\left(T_i,\,T_{i+1}\right)}-\left(1+\alpha_{T_i,T_{i+1}}L_x\right)\right)^+\right].
\end{aligned}$$

Option pricing in HJM

- Under the T_{i+1} measure, the quantity $Q(t, T_i, T_{i+1}) = P(t, T_i)/P(t, T_{i+1})$ is a relative price and then a martingale:

$$\frac{P(t, T_i)}{P(t, T_{i+1})} = \mathbb{E}_t^{T_{i+1}}\left[\frac{P(T_i, T_i)}{P(T_i, T_{i+1})}\right]$$

- In HJM Gaussian models, $Q(t, T_i, T_{i+1})$, being a ratio of lognormal prices, is lognormal (whilst in HJM the simple forward rate is not lognormal!!).
- We can apply the Black formula to $Q(t, T_i, T_{i+1})$.



The caplet formula for HJM

- Considering as underlying variable the quantity $Q(t, T_i, T_{i+1})$, we can apply the Black formula:

$$
\begin{aligned}
& P(t, T_{i+1}) \left[Q(t, T_i, T_{i+1}) \mathcal{N}(d_1) - (1 + \tau L_x) \mathcal{N}(d_2) \right] \qquad (19) \\
= \; & P(t, T_{i+1}) \left[\left(1 + \tau F_{t, T_i, T_{i+1}}\right) \mathcal{N}(d_1) - (1 + \tau L_x) \mathcal{N}(d_2) \right]
\end{aligned}
$$

where:

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{(1+\tau F(t, T_i, T_{i+1}))}{(1+\tau L_x)} + \frac{1}{2} \Sigma^2_{t, T_i, T_{i+1}}}{\sqrt{\Sigma^2_{t, T_i, T_{i+1}}}}; \\
d_2 &= d_1 - \sqrt{\Sigma^2_{t, T_i, T_{i+1}}} \\
\Sigma^2_{t, T_i, T_{i+1}} &= \int_t^{T_i} \mathbb{V}ar_s \left(d \ln Q(s, T_i, T_{i+1}) \right) \\
&= \int_t^{T} \left(\sigma_P(u, T_i) - \sigma_P(u, T_{i+1}) \right)^2 du
\end{aligned}
$$

$\Sigma^2_{t,T_i,T_{i+1}}$

- $\Sigma^2_{t,T_i,T_{i+1}}$ represents the integrated instantaneous variance of $d\ln Q$ and depends on the particular Gaussian HJM model we are using.
- Ho and Lee
$$\Sigma^2(t, T, T+\tau) = \sigma^2\tau^2(T-t).$$
- Extended Vasicek Model
$$\Sigma^2(t, T, T+\tau) = \frac{\sigma^2}{2\lambda^3}\left(1-e^{-\lambda\tau}\right)^2\left(1-e^{-2\lambda(T-t)}\right).$$
- Two factors Hull and White model (see Brigo and Mercurio pag. 151):
$$\begin{aligned}
&\Sigma^2(t, T, T+\tau)\\
&= \frac{\sigma_1^2}{2\lambda_1^3}\left(1-e^{-\lambda_1\tau}\right)^2\left(1-e^{-2\lambda_1(T-t)}\right)\\
&+\frac{\sigma_2^2}{2\lambda_2^3}\left(1-e^{-\lambda_2\tau}\right)^2\left(1-e^{-2\lambda_2(T-t)}\right)\\
&+2\rho\frac{\sigma_1\sigma_2}{\lambda_1+\lambda_2}\frac{\left(-e^{-\lambda_1\tau}+1\right)}{\lambda_1}\frac{\left(-e^{-\lambda_2\tau}+1\right)}{\lambda_2}\left(1-e^{-(\lambda_1+\lambda_2)(T-t)}\right).
\end{aligned}$$

The Forward LIBOR Market Model

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Main References

Useful Readings

- Brigo Damiano and Fabio Mercurio, Interest Rate Models: Theory and Practice, Springer Finance 2001.
- Pietro Veronesi. Fixed Income Securities. **Chapter 21.4 & 22.6.**
- J. C. Hull, Options, Futures, and Other Derivatives, Global Edition, Pearson Education M.U.A., Published: 16 June, 2017 **Chapters 28, 29, 30** .
- Antoon Pelsser, Efficient Methods for Valuing Interest Rate Derivatives, Springer-Verlag London(2000), **Chapter: 8, LIBOR and Swap Market Models**.

Accompanying Excel File

- FI_LMM_MC.xlsx

Outline I

1. **Forward rate dynamics under the same measure**

2. **The FLMM & Monte Carlo simulation**
   - A Numerical Example: Simulating the Forward Curve
   - Pricing a cap
   - Pricing a swaption

How many forward measures?

- If we use the forward measure, the forward rate $F(t, T_i, T_{i+1})$ is a martingale only under the measure T_{i+1}.
- So each forward rate is a martingale under a particular measure.
- In order to price more complex instruments, we need to model together rates of different maturities under the same measure (e.g. the risk neutral one, or the so called terminal measure).
- In order to reconduct the forward rate processes under the same measure, we need a change of measure and the Girsanov theorem.
- Forward Libor rates considered under different measures, will be anymore martingale so that expensive Monte Carlo simulation will be required.

The Libor Market Model: Forward rates under the same measure

- The LIBOR market model is set up by assuming a lognormal dynamics for each forward LIBOR rate $F_i(t) = F(t, T_i, T_{i+1})$ with respect to the probability measure $\mathbb{Q}^{i+1}$.
- Hence each LIBOR rate is modelled under a different probability measure. However, to price exotic derivatives, we need to model all LIBOR rates under the same measure.
- The sde's for two adjacent forward rates $F_{i-1}(t)$ and $F_i(t)$, under their respective measures, are:

log-Martingale

$$\begin{aligned} dF_{i-1}(s) &= \sigma_{i-1}(s) F_{i-1}(s) dW^i, \\ dF_i(s) &= \sigma_i(s) F_i(s) dW^{i+1}. \end{aligned}$$

- Here, we assume that the two forward rates are perfectly correlated, but we can generalize to a non-perfect correlation case.

$F_{i-1}(t)$ under $\mathbb{Q}^{i+1}$ I

- Our problem is to find the dynamics of $F_{i-1}(t)$ under the measure $\mathbb{Q}^{i+1}$.
- It is convenient in the derivation, to move from the $\mathbb{Q}^{i+1}$ measure to the $\mathbb{Q}^{i}$.
- Let us consider the ratio of numeraires that gives us the R-N derivative $\psi(T)$:

$$\psi(T) = \frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{P(T,T_i)}{P(t,T_i)}\frac{P(t,T_{i+1})}{P(T,T_{i+1})} \quad (1)$$

$$= \frac{1+F(T,T_i,T_{i+1})\,\alpha_{i,i+1}}{1+F(t,T_i,T_{i+1})\,\alpha_{i,i+1}} \quad (2)$$

- In order to use the Girsanov Theorem we need to find $k(t)$ s.t.

$$\psi(T) = \exp\left(-\frac{1}{2}\int_t^T k^2(s)\,ds + \int_t^T k(s)\,dW^{i+1}(s)\right).$$

- We recall that $\psi(s)$, $t < s < T$, is martingale under the original measure $\mathbb{Q}^{i+1}$. Moreover, by applying Ito's lemma, we get

$$d\psi(s) = 0 \times ds + k(s)\,\psi(s)\,dW^{i+1}(s). \quad (3)$$

$F_{i-1}(t)$ under $\mathbb{Q}^{i+1}$ II

- An application of Ito's lemma to (2), gives us for $s > t$:

$$\begin{aligned} d\psi(s) &= \frac{1}{1+F_i(t)\,\alpha_{i,i+1}} d\left(1+F_i(s)\,\alpha_{i,i+1}\right) && (4)\\ &= \frac{1}{1+F_i(t)\,\alpha_{i,i+1}} \sigma_i(s)\,F_i(s)\,\alpha_{i,i+1} dW^{i+1}(s) && (5)\\ &= \frac{1+F_i(s)\,\alpha_{i,i+1}}{1+F_i(t)\,\alpha_{i,i+1}} \frac{\sigma_i(s)\,F_i(s)\,\alpha_{i,i+1}}{1+F_i(s)\,\alpha_{i,i+1}} dW^{i+1}(s) && (6)\\ &= \psi(s)\,\frac{\sigma_i(s)\,F_i(s)\,\alpha_{i,i+1}}{1+F_i(s)\,\alpha_{i,i+1}} dW^{i+1}(s) && (7) \end{aligned}$$

- Comparing 3 and 7, we can identify k as:

$$\rightarrow k(s) = \frac{\sigma_i(s)\,F_i(s)\,\alpha_{i,i+1}}{1+F_i(s)\,\alpha_{i,i+1}}.$$

$F_{i-1}(t)$ under $\mathbb{Q}^{i+1}$ III

- Girsanov's Theorem now gives the relation:

$$dW^{i}(s) = dW^{i+1}(s) - k(s)\,ds$$

and the process for $F_{i-1}(s)$ under the measure $\mathbb{Q}^{i+1}$ will be:

$$\begin{aligned} dF_{i-1}(s) &= \sigma_{i-1}(s)\,F_{i-1}(s)\,dW^{i} \\ &= -\frac{\sigma_{i-1}(s)\,F_{i-1}(s)\,\sigma_i(s)\,F_i(s)\,\alpha_{i,i+1}}{1+F(s,T_i,T_{i+1})\,\alpha_{i,i+1}}ds \\ &\quad +\sigma_{i-1}(s)\,F_{i-1}(s)\,dW^{i+1}(s)\,. \end{aligned}$$

$F_{i-1}(t)$ under $\mathbb{Q}^{i+1}$ IV

- If we use repeatedly this result, we can obtain that for $t \leq s \leq T_i$, $F_i(s)$ has sde under the terminal measure $\mathbb{Q}^{N+1}$

$$
\begin{aligned}
dF_i(s) &= -\sum_{k=1+i}^{N} \frac{\sigma_i(s) F_i(s) \sigma_k(s) F_k(s) \alpha_{k,k+1}}{1 + F(s, T_k, T_{k+1}) \alpha_{k,k+1}} dt \\
&\quad + \sigma_i(s) F_i(s) dW^{N+1}(s).
\end{aligned}
$$

- Therefore, apart from $F_N(t)$, all forward Libor rates are no longer martingales under the terminal measure, but have a drift that depends on the forward Libor rates with longer maturities.

Remarks

1. The set of sde for $i = 1, \cdots, N$ represents the Forward LIBOR Market Model.
2. The implementation can be performed via Monte Carlo simulation, because we need to simulate all forward rates at the same time.
3. The model calibration is straightforward: the volatilities σ_i are obtained by bootstrapping the term structure of volatilities of cap prices. Unfortunately, we cannot fit the full smile and for this we need stochastic volatility models.
4. If we assume that the Brownian motions driving the F_j and F_i forward rate are correlated with correlation ρ_{ij}, the dynamics becomes

$$
\begin{aligned}
dF_i(s) &= -\sum_{k=1+i}^{N} \frac{\sigma_i(s) F_i(s) \sigma_k(s) F_k(s) \alpha_{k,k+1}}{1 + F(s, T_k, T_{k+1}) \alpha_{k,k+1}} \boldsymbol{\rho}_{i,k} dt \\
&\quad + \sigma_i(s) F_i(s) dW_i^{N+1}(s).
\end{aligned}
$$

5. The Bachelier version is obtained replacing everywhere $\sigma_i F_i$ by σ_i.
6. Brigo and Mercurio, pagg. 198-203, discuss also the problems of modelling $F_i(t)$ under the risk neutral measure, that is required for some product like Eurodollar futures.

The FLMM & Monte Carlo simulation

Monte Carlo simulation of forward rates I

1. Let us consider the dynamics of forward rates $F_i(t) = F(t, T_i, T_{i+1}), i = 0, \cdot, N$ under the T_{N+1} measure (assume perfect correlations between Brownian motions)

$$
\begin{aligned}
dF_i(s) &= -\sum_{k=1+i}^{N} \frac{\sigma_i(s) F_i(s) \sigma_k(s) F_k(s) \alpha_{k,k+1}}{1 + F(s, T_k, T_{k+1}) \alpha_{k,k+1}} dt \\
&\quad + \sigma_i(s) F_i(s) dW^{N+1}(s).
\end{aligned}
$$

2. Apart from $F_N(t)$ all the remaining forward rates are not martingale under the terminal measure, but have a drift that depends on the values of the forward rates with a longer maturity.
3. In general, these set of sde's do not admit a closed form solution, so we have to use numerical methods such as Monte Carlo method to solve it.

Monte Carlo simulation of forward rates II

4. Set $\alpha_i = T_{i+1} - T_i$ to be the tenor of the forward rate and Δ the time step in the time-discretization.
5. It is convenient to set the time step dt equal to the tenor of the forwar rates. So if we evolve monthly (quarterly/semi-annual/annual) forward rates we adopt a monthly (quarterly/semi-annual/annual) step.
6. The increment of the Brownian motions are simulated via

$$dW^{N+1} = \epsilon_i \sqrt{\Delta}$$

where $\epsilon_i \sim \mathcal{N}(0,1)$.

7. We also need a term structure of volatilities

$$\sigma_{j\Delta}(s) = StDev(dF(s, s + j\Delta, s + (j+1)\Delta))$$

that can be obtained via the bootstrapping of the term structure of cap volatilities. Notice that we also need to specify the time evolution (with respect to time s) of this term structure. Different assumptions are described in Brigo and Mercurio.

Monte Carlo simulation of forward rates III

8. We also discretize the sde of $log(F_i(t))$, so we are sure that forward rates remain positive.

$$F_i(s+\Delta) = F_i(s) e^{\left(-\sum_{k=1+i}^{N} \frac{\sigma_i(s)\sigma_k(s)F_k(s)\alpha_{k,k+1}}{1+F(s,T_k,T_{k+1})\alpha_{k,k+1}} - \frac{\sigma_i^2(s)}{2}\right)dt + \sigma_i(s)dW_i^{N+1}(s)},$$

for $s = 0, \Delta, \cdots, N\Delta$.   assume: $\sigma_i(s) = \sigma_i(0) \ \forall t$

Table: Simulation scheme of the term structure of forward rates with tenor Δ. The terminal measure is the 5Δ forward measure. Simulation ends at time 4Δ

Start End		0 Δ	Δ 2Δ	2Δ 3Δ	3Δ 4Δ	4Δ 5Δ
Time Step	dW	Tenor Vol. TS	Δ σ_Δ	Δ $\sigma_{2\Delta}$	Δ $\sigma_{3\Delta}$	Δ $\sigma_{4\Delta}$
0		$F(0,0,\Delta)$	$F(0,\Delta,2\Delta)$	$F(0,2\Delta,3\Delta)$	$F(0,3\Delta,4\Delta)$	$F(0,4\Delta,5\Delta)$
Δ	$\epsilon_1\sqrt{\Delta}$		$F(\Delta,\Delta,2\Delta)$	$F(\Delta,2\Delta,3\Delta)$	$F(\Delta,3\Delta,4\Delta)$	$F(\Delta,4\Delta,5\Delta)$
2Δ	$\epsilon_2\sqrt{\Delta}$			$F(2\Delta,2\Delta,3\Delta)$	$F(2\Delta,3\Delta,4\Delta)$	$F(2\Delta,4\Delta,5\Delta)$
3Δ	$\epsilon_3\sqrt{\Delta}$				$F(3\Delta,3\Delta,4\Delta)$	$F(3\Delta,4\Delta,5\Delta)$
4Δ	$\epsilon_4\sqrt{\Delta}$					$F(4\Delta,4\Delta,5\Delta)$

Monte Carlo simulation A Numerical Example

Example (MC Simulation of the Forward LIBOR market model)

- Assume the initial forward curve with 6m tenor is flat at 2%. The time step is semi-annual. Assume the volatility term structure is flat at 20%.
- **[Step 1]** The simulated increment of the Brownian motion over the first time step is -0.1356.
- We obtain that $F(0.5, 0.5, 1) =$

$$2\% e^{\left(-\left(\frac{0.2\times0.2\times0.02\times0.5}{1+0.02\times0.5}+\frac{0.2\times0.2\times0.02\times0.5}{1+0.02\times0.5}+\frac{0.2\times0.2\times0.02\times0.5}{1+0.02\times0.5}\right)-\frac{0.2^2}{2}\right)\times0.5+0.2\times(-0.1356)} = 1.926\%$$

- We obtain that $F(0.5, 1, 1.5) =$

$$2\% e^{\left(-\left(\frac{0.2\times0.2\times0.02\times0.5}{1+0.02\times0.5}+\frac{0.2\times0.2\times0.02\times0.5}{1+0.02\times0.5}\right)-\frac{0.2^2}{2}\right)\times0.5+0.2\times(-0.1356)} = 1.9264\%$$

- We obtain that $F(0.5, 1.5, 2) =$

$$2\% e^{\left(-\left(\frac{0.2\times0.2\times0.02\times0.5}{1+0.02\times0.5}\right)-\frac{0.2^2}{2}\right)\times0.5+0.2\times(-0.1356)} = 1.9267\%$$

- We obtain that $F(0.5, 2, 2.5) =$

$$2\% e^{-\frac{0.2^2}{2}\times0.5+0.2\times(-0.1356)} = 1.9271\%$$

Example ((..ctd))

- The term structure of discount factors at time s is given by

$$P(s, s + j\Delta) = P(s, s + (j-1)\Delta)\frac{1}{1 + F(s, s + (j-1)\Delta, s + j\Delta)\Delta},$$

starting with $P(s, s + \Delta) = 1/(1 + F(s, s, s + \Delta)\Delta)$.

- At the initial time, we have

$$P(0, \Delta) = 99.0099\%; \quad P(0, 2\Delta) = 98.0296\%;$$

$$P(0, 3\Delta) = 97.0590\%; \quad P(0, 4\Delta) = 96.0980\%; \quad P(0, 5\Delta) = 95.1466\%.$$

- After 6m the term structure of discount factors is given by

$$P(0.5, 1) = \frac{1}{1+1.926\%\times 0.5} = 99.0462\%,$$
$$P(0.5, 1.5) = \frac{99.0462\%}{1+1.9264\%\times 0.5} = 98.101\%,$$
$$P(0.5, 2) = \frac{98.101\%}{1+1.9267\%\times 0.5} = 97.165\%,$$
$$P(0.5, 2.5) = \frac{97.165\%}{1+1.9271\%\times 0.5} = 96.238\%,$$

Example ((..ctd))

- **[Step 2]** The simulated increment of the Brownian motion over the second time step is -0.1899.
- We obtain that $F(1,1,1.5) =$

$$1.9264\% e^{\left(-\left(\frac{0.2\times0.2\times0.019267\times0.5}{1+0.019267\times0.5}+\frac{0.2\times0.2\times0.019271\times0.5}{1+0.019271\times0.5}\right)-\frac{0.2^2}{2}\right)\times0.5+0.2\times(-0.1899)} = 1.835\%$$

- In addition,

$$F(1,1.5,2) = 1.9267\% e^{\left(-\frac{0.2\times0.2\times0.019271\times0.5}{1+0.019271\times0.5}-\frac{0.2^2}{2}\right)\times0.5+0.2\times(-0.1899)} = 1.836\%$$

and

$$F(1,2,2.5) = 1.9271\% e^{-\frac{0.2^2}{2}\times0.5+0.2\times(-0.1899)} = 1.8371\%$$

- The simulated discount curve in 1 year time is

$$P(1,1.5) = \frac{1}{1+1.835\%\times0.5} = 99.091\%,$$
$$P(1,2) = \frac{99.091\%}{1+1.836\%\times0.5} = 98.189\%,$$
$$P(1,2.5) = \frac{98.189\%}{1+1.8371\%\times0.5} = 97.296\%.$$

Example ((..ctd))

- **[Step 3]** The simulated increment of the Brownian motion over the third time step is 0.0984.
- We obtain that $F(1.5, 1.5, 2) =$

$$1.836\% e^{\left(-\frac{0.2\times0.2\times0.018371\times0.5}{1+0.018371\times0.5} - \frac{0.2^2}{2}\right)\times0.5+0.2\times(0.0984)} = 1.854\%$$

- In addition,

$$F(1.5, 1.5, 2.5) = 1.8371\% e^{-\frac{0.2^2}{2}\times0.5+0.2\times(0.0984)} = 1.855\%$$

- The simulated discount curve in 1.5 year is

$$P(1.5, 2) = \frac{1}{1+1.854\%\times0.5} = 99.082\%, \quad P(1.5, 2.5) = \frac{99.082\%}{1+1.855\%\times0.5} = 98.171\%,$$

- **[Step 4]** The simulated increment of the Brownian motion over the fourth time step is -0.5587 and

$$F(2, 2, 2.5) = 1.855\% e^{\left(-\frac{0.2^2}{2}\right)\times0.5+0.2\times(-0.5587)} = 1.642\%$$

and the discount factor is $P(2, 2.5) = 1/(1 + 1.642\% \times 0.5) = 99.186\%$.

Table: Simulated forward and discount curve

		Forward Rates				
Start		0.00	0.50	1.00	1.50	2.00
End		0.50	1.00	1.50	2.00	2.50
Tenor		Tenor	0.5	0.5	0.5	0.5
Time	**BM**	Vol. TS	20.00%	20.00%	20.00%	20.00%
0.00		2.00%	2.00%	2.00%	2.00%	2.00%
0.50	-0.13560		1.9260%	1.9264%	1.9267%	1.9271%
1.00	-0.18990			1.8354%	1.8361%	1.8368%
1.50	0.09840				1.8536%	1.8547%
2.00	-0.55870					1.6421%
Time	Discount Factors					
0.00	100%	99.0099%	98.0296%	97.0590%	96.0980%	95.1466%
0.50		100%	99.0462%	98.1013%	97.1652%	96.2379%
1.00			100%	99.0907%	98.1892%	97.2956%
1.50				100%	99.0817%	98.1713%
2.00					100%	99.1856%

Example (Pricing a CAP)

- We price a cap with 2 years tenor and strike 1.5% with 6m LIBOR as reference rate. It contains the following caplets 6mx12m, 12mx18m, 18mx24m.
- Reset dates are: 6m, 12m and 18m; Payment dates are 12m, 18m and 24m.
- We use the 2.5-forward measure, i.e. the numeraire is the zcb expiring in 2.5 years.
- Given the simulated forward rates we obtain the simulated 6m LIBOR rates and one simulated relative payoff of the cap that is 0.5738%.

Pricing a CAP with MC Simulation

Reset	Payment	6mLIBOR	Strike	Payoff	Numeraire	Payoff/Numeraire
0.5	1	1.9260%	1.5000%	0.2130%	97.2956%	0.2189%
1	1.5	1.8354%	1.5000%	0.1677%	98.1713%	0.1708%
1.5	2	1.8536%	1.5000%	0.1768%	99.1856%	0.1783%
					Sum	**0.5680%**

Example ((...ctd))

We repeat the simulation 100,000 times and we obtain the MC estimate of the cap as

$$cap = 95.1466\% \times \text{Average Payoff}$$

where Average Payoff is given by

$$\frac{1}{100000}\sum_{i=1}^{100,000}\left(\frac{caplet_i(6mx12m)}{P_i(1,2.5)} + \frac{caplet_i(12mx18m)}{P_i(1.5,2.5)} + \frac{caplet_i(18mx24m)}{P_i(2,2.5)}\right)$$

Simulation	MC
1	0.7466%
2	0.0562%
3	0.3769%
...	0.6670%
...	0.5542%
...	0.1441%
7	0.9784%
8	0.3718%
9	0.8726%
100,000	0.3580%
Avg Payoff	0.5126%
Numeraire	95.1466%
CAP Price	**0.4877%**

Example (Pricing a swaption)

- We price a swaption expiring in 12 m and strike 1.5%.
- It is written on swap having 12m tenor with semi-annual payments.
- We price it by MC simulation computing using the zcb expiring in 2.5 years as numeraire

$$P(0,2.5) \times \frac{1}{M} \sum_{i=1}^{M} \frac{(S_i(12m) - 1.5\%)^+ \times Annuity_i(12m)}{P_i(12m, 30m)}$$

where i refers to the simulation.

- Using the simulated values we have

Pricing a SWAPTION with MC Simulation

Expiry	Annuity	Floating Leg	FSR	Strike	Payoff	Numeraire	Payoff/Numeraire
1	0.97742	1.80%	1.84%	1.50%	0.003289	97.2956%	0.003380

A. $Annuity = 0.5 \times (98.1892\% + 97.2956\%) = 97.742\%$;

B. $FloatingLeg = 99.0907\% - 97.2956\% = 1.80\%$;

C. $ForwardSwapRate = \frac{1.80\%}{97.742\%} = 1.84\%$;

D. $Payoff = (1.84\% - 1.5\%)^+ \times 97.742\% = 0.003289$;

E. $\frac{Payoff}{Numeraire} = \frac{0.003289}{97.2956\%} = 0.003380.$

Example ((...ctd))

We repeat the simulation 100,000 times and we obtain the MC estimate of the swaption as

$$cap = 95.1466\% \times \text{Average Payoff}$$

where Average Payoff is given by

$$\frac{1}{M}\sum_{i=1}^{M}\frac{(S_i(12m) - 1.5\%)^{+} \times Annuity_i(12m)}{P_i(12m, 30m)}$$

NR. Simulation	Payoff/Numeraire
1	0.2904%
2	0.1266%
3	0.4975%
...	0.9891%
...	0.8667%
...	0.2880%
7	0.3671%
8	0.6620%
9	0.2754%
100,000	0.9590%
Avg	0.5322%
Numeraire	95.1466%
Price	**0.5063%**

Conclusions

- We have presented the Forward LIBOR market model.
- We have illustrated how to implement a Monte Carlo simulation.
- The required inputs are

A. the initial term structure of simple forward rates with a given tenor; It is obtainted by boostrapping market quotations of LIBOR and swap rates

B. the initial term structure of volatilities (fwd volatilities) of simple forward rates with a given tenor; It is obtainted by boostrapping market quotations of cap volatilities. We need to specify the time evolution of this term structure. See Brigo and Mercurio for different specifications.

C. In order to have a multifactor model, we also need the correlations among forward rates. These can be estimated using historical time series or implied by quotations of the implied volatilities of swaptions.

- The presented MC simulation can be used also for pricing more complex interest rate derivatives.